Local and global bifurcation of electron-states

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Abstract

We study the bifurcation of traveling periodic electron layers, that we call *electron-states*, from symmetric and asymmetric flat velocity strips in the phase space, for the one dimensional Vlasov-Poisson equation with space periodic condition. The boundaries of the constructed solutions are real-analytic and in uniform translation at the same speed in the space direction. These structures are obtained applying Crandall-Rabinowitz's Theorem using either the velocity or geometrical quantities related to the size of the strip as bifurcation parameters. In the first case, we can prove for any fixed symmetry, the emergence of a pair of branches and the local bifurcation diagram has a hyperbolic structure. In the symmetric situation, we find, for any large enough symmetry, one branch whose orientation close to the stationary solution depends on the sign of the prescribed speed of translation. As for the asymmetric case, we find either a countable or a finite number of bifurcation curves according to some constraints related to the prescribed speed of translation. The pitchfork (subcritical or supercritical) bifurcation is also described in this case. Finally, we briefly discuss the global continuation of these branches.

Contents

1 Intoduction

We present here the equation studied in this work which is a kinetic model in dimension one with space periodic boundary conditions. Then, we discuss a particular class of weak solutions to this equation called *electron layers* that are renormalized characteristic functions of time and space dependent velocity domains. The velocity flat strips provide stationary solutions and we present some pertubative existence results of time periodic solutions close to these equilibrium states.

1.1 One dimensional Vlasov-Poisson equation and patches of electrons

We consider the 1D Vlasov-Poisson equation with space 1-periodic boundary condition

$$\partial_t f(t,x,v) + v\,\partial_x f(t,x,v) - E(t,x)\,\partial_v f(t,x,v) = 0, \qquad (t,x,v) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}. \tag{1.1}$$

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Here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the flat torus that we liken to the segment $[0,1]$ where 0 and 1 are identified. The equation (1.1) is a model that can be found in [2, Chap. 13] or [13]. It describes a collisionless neutral plasma composed with ions and electrons. The ions' significant inertia enables us to consider them as a neutralizing uniform background field. The unknown $f(t,x,v)$ represents the density of electrons traveling with speed v at position x and time t. We make the assumption that the plasma properties are one-dimensional. Hence, the transport is unidirectional and the problem is simplified to one space dimension. In particular, the particle motion is only influenced by induced electrostatic forces and therefore we disregard electromagnetic interactions. The electric field E is associated to the electric potential $\boldsymbol{\varphi}$ as follows

$$E(t,x) = \partial_x \boldsymbol{\varphi}(t,x), \qquad \partial_{xx}\boldsymbol{\varphi}(t,x) = 1 - \int_{\mathbb{R}} f(t,x,v)dv. \tag{1.2}$$

According to (1.2) and Taylor formula, the periodic constraint

$$E(t,0) = E(t,1)$$

is equivalent to the neutrality condition

$$\int_0^1 \int_{\mathbb{R}} f(t,x,v)dxdv = 1. \tag{1.3}$$

Observe that in the problem the quantity of interest is $\partial_x\boldsymbol{\varphi}$. Then $\boldsymbol{\varphi}$ is defined up to a time dependent additive constant that we can choose in order to impose, for any time, a zero space average condition for $\boldsymbol{\varphi}$. As a consequence, introducing the inverse Laplace operator ∂_{xx}^{-1} defined as follows

$$\forall j \in \mathbb{Z}^*, \quad \partial_{xx}^{-1}\mathbf{e}_j = \frac{-\mathbf{e}_j}{4\pi^2 j^2}, \qquad \mathbf{e}_j(x) \triangleq e^{2i\pi jx},$$

we get from (1.2)-(1.3)

$$\boldsymbol{\varphi}(t,x) = \partial_{xx}^{-1}\left(1 - \int_{\mathbb{R}} f(t,x,v)dv\right). \tag{1.4}$$

The equation (1.1) can recast as an active scalar equation. To this aim, we see the phase space $\mathbb{T}\times\mathbb{R}$ as a cylinder manifold embedded in $\mathbb{R}^3$ with radius $r=1$ and with vertical axis soul. The identification can be done through the local chart

$$\begin{array}{rcl} (0,1)\times\mathbb{R} & \to & \mathbb{R}^3 \\ (x,v) & \mapsto & \big(\cos(2\pi x), \sin(2\pi x), v\big). \end{array}$$

At any point $(x,v)\in\mathbb{T}\times\mathbb{R}$, the tangent plane $T_{(x,v)}(\mathbb{T}\times\mathbb{R}) \equiv \mathbb{R}^2$ admits the orthonormal basis (with the classical identification vector/directional derivative)

$$\mathsf{e}_x \triangleq \partial_x, \qquad \mathsf{e}_v \triangleq \partial_v.$$

For any function $\mathsf{g} : \mathbb{T}\times\mathbb{R}\to\mathbb{R}$, the gradient is given by

$$\nabla_{x,v}\mathsf{g}(x,v) = \partial_x\mathsf{g}(x,v)\mathsf{e}_x + \partial_v\mathsf{g}(x,v)\mathsf{e}_v.$$

The orthogonal gradient is obtained by a rotation of angle $\frac{\pi}{2}$

$$\nabla^{\perp}_{x,v} \triangleq \mathtt{J}_{x,v}\nabla_{x,v}, \qquad \underset{(\mathsf{e}_x,\mathsf{e}_v)}{\mathrm{Mat}}\,(\mathtt{J}_{x,v}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Consider the velocity field

$$\begin{array}{rrcl} \mathbf{v}: & \mathbb{T}\times\mathbb{R} & \to & T(\mathbb{T}\times\mathbb{R}) \triangleq \displaystyle\bigcup_{(x,v)\in\mathbb{T}\times\mathbb{R}} T_{(x,v)}(\mathbb{T}\times\mathbb{R}) \\ & (x,v) & \mapsto & v\mathsf{e}_x - E(t,x)\mathsf{e}_v, \end{array}$$

which is divergence-free

$$\mathrm{div}_{x,v}\mathbf{v}(t,x,v) = \partial_x(v) + \partial_v\big(-E(t,x)\big) = 0. \tag{1.5}$$

More precisely, we can write

$$\mathbf{v}(t,x,v) = -\nabla^{\perp}_{x,v}\boldsymbol{\Psi}(t,x,v), \qquad \boldsymbol{\Psi}(t,x,v) \triangleq \frac{v^2}{2} + \boldsymbol{\varphi}(t,x). \tag{1.6}$$

Then, the equation (1.1) becomes

$$\partial_t f(t,x,v) + \Big\langle \mathbf{v}(t,x,v)\,,\nabla_{x,v} f(t,x,v)\Big\rangle_{T_{(x,v)}(\mathbb{T}\times\mathbb{R})} = 0, \tag{1.7}$$

where the scalar product defined by

$$\Big\langle \alpha(x,v)\mathsf{e}_x + \beta(x,v)\mathsf{e}_v\,, \gamma(x,v)\mathsf{e}_x + \delta(x,v)\mathsf{e}_v\Big\rangle_{T_{(x,v)}(\mathbb{T}\times\mathbb{R})} \triangleq \alpha(x,v)\gamma(x,v) + \beta(x,v)\delta(x,v).$$

The global existence of classical solutions to (1.1) was discussed by Cottet-Raviart [8] and the existence of periodic mild solutions has been studied by Bostan-Poupaud [3]. In his thesis [13, Thm. 2.1.1], Dziurzynski proved that any initial datum $f_0 \in L^\infty(\mathbb{T}\times\mathbb{R})$ with compact support satisfying (1.3) generates a unique global in time weak solution $f \in L^\infty\big([0,\infty), L^\infty(\mathbb{T}\times\mathbb{R})\big)$ which is Lagrangian, namely

$$f(t,x,v) = f_0\big(X_t^{-1}(x,v)\big),$$

where X_t is the flow map associated with the velocity field $\mathbf{v}$ and given by

$$\partial_t X_t(x,v) = \mathbf{v}\big(t, X_t(x,v)\big), \qquad X_0(x,v) = (x,v).$$

In particular, if we consider a bounded initial domain Ω_0 and set $\Omega_t \triangleq X_t(\Omega_0)$, then

$$f(t,x,v) = \frac{1}{|\Omega_t|}\mathbf{1}_{\Omega_t}(x,v)$$

is a weak solution called *patch of electrons.* In addition, the divergence-free condition (1.5) implies the conservation of the area, that is $|\Omega_t| = |\Omega_0|$. We mention that patches of electrons do not physically model a concentration of electrons since the domain is in the phase space $\mathbb{T}\times\mathbb{R}$. Patches of electrons are the kinetic equivalent of the vortex patches in fluid mechanics. We refer the reader to [2, 41] for a general introduction to the vortex patch dynamics. In the sequel, we shall work with a subclass of patches of electrons called *electron layers* and defined as follows. Consider an initial condition with strip-shaped domain

$$f_0(x,v) = \frac{1}{|S_0|}\mathbf{1}_{S_0}(x,v), \qquad S_0 \triangleq \big\{(x,v)\in\mathbb{T}\times\mathbb{R}, \quad \text{s.t.} \quad v_-^0(x) < v < v_+^0(x)\big\}$$

associated with two initial periodic profiles $v_\pm^0$. Then, the corresponding weak solution writes

$$f(t,x,v) = \frac{1}{|S_t|}\mathbf{1}_{S_t}(x,v), \qquad S_t \triangleq X_t(S_0).$$

At later time $t>0$, the domain S_t is still a velocity strip, that is one can find two periodic profiles $x \mapsto v_\pm(t,x)$ such that

$$S_t = \big\{(x,v)\in\mathbb{T}\times\mathbb{R} \quad \text{s.t.} \quad v_-(t,x) < v < v_+(t,x)\big\}.$$

With these notations, the area-preserving condition writes

$$\int_0^1 \big[v_+(t,x) - v_-(t,x)\big]dx = |S_t| = |S_0| = \int_0^1 \big[v_+^0(x) - v_-^0(x)\big]dx. \tag{1.8}$$

Dziurzynski also showed in [13] the global in time persistence for the C^1 regularity of the boundary ∂S_t. In addition, he numerically exposed possible folding formation. In this latter case, he proved possible loss of C^3-smoothness in finite time and excluded the formation of cusps. In the next lemma, we provide a family of electron layer stationary solutions parametrized by two real numbers $a<b$ related to the geometry of the patch.

Lemma 1.1. *For any $(a,b)\in\mathbb{R}^2$ with $a<b$, the initial profile*

$$f_0(x,v) = \frac{1}{b-a}\mathbf{1}_{S_{\text{flat}}(a,b)}(x,v), \qquad S_{\text{flat}}(a,b) \triangleq \mathbb{T}\times[a,b] \tag{1.9}$$

generates a stationary electron layer.

Proof. Let $a<b$ and consider a function f in the form

$$f(t,x,v) = f_0(x,v) = \frac{1}{b-a}\mathbf{1}_{a\leqslant v\leqslant b}.$$

Then one has $\partial_t f = \partial_x f = 0$. Besides, the identity (1.4) together with the structure of f and the neutrality condition (1.3) imply $\boldsymbol{\varphi} = 0$. Thus $E=0$ and f solves (1.1). ☐

Remark 1.1. *More generally, the previous proof shows that any function of the variable v only is a stationary solution of* (1.1)*. This is a classical result in kinetic theory.*

1.2 Perturbative approach for the electron layer dynamics

The scope of this subsection is to obtain the equations of motion for a general electron layer. Then, introducing small deformations of the flat strip $S_{\text{flat}}(a,b)$ with $a<b$, we prove that they are solutions to a system of two coupled quasilinear transport equations with linear coupling, see (1.23).

Due to the transport structure (1.7), the dynamics is entirely characterized by the evolution of the boundaries. We provide here the complete derivation of the contour dynamics equations following the general computations in [27, Sec. 3.1] but adapted to our notations. We denote Γ_0^+ and Γ_0^- the two boundaries of the initial strip S_0. They can be seen as the zero level sets of two C^1 regular functions g_0^+ and g_0^- from $\mathbb{T}\times\mathbb{R}$ into $\mathbb{R}$, namely

$$\Gamma_0^{\pm}=\big\{(x,v)\in\mathbb{T}\times\mathbb{R}\quad\text{s.t.}\quad g_0^{\pm}(x,v)=0\big\},\qquad\forall(x,v)\in\mathbb{T}\times\mathbb{R},\quad\nabla_{x,v}\,g_0^{\pm}(x,v)\neq0.$$

We set

$$g_{\pm}(t,x,v)\triangleq g_0^{\pm}\big(X_t^{-1}(x,v)\big),\qquad\text{i.e.}\qquad g_{\pm}\big(t,X_t(x,v)\big)\triangleq g_0^{\pm}(x,v).\tag{1.10}$$

By construction, for any time t, the boundaries Γ_t^+ and Γ_t^- of S_t are the zero level sets of the functions $g_+(t,\cdot,\cdot)$ and $g_-(t,\cdot,\cdot)$, respectively

$$\Gamma_t^{\pm}\triangleq X_t\big(\Gamma_0^{\pm}\big)=\big\{(x,v)\in\mathbb{T}\times\mathbb{R}\quad\text{s.t.}\quad g_{\pm}(t,x,v)=0\big\}.$$

Differentiating in time the relation (1.10), we get

$$\begin{aligned}0&=\partial_t g_{\pm}\big(t,X_t(x,v)\big)+\Big\langle\partial_t X_t(x,v)\,,\nabla_{x,v}\,g_{\pm}\big(t,X_t(x,v)\big)\Big\rangle_{T_{(x,v)}(\mathbb{T}\times\mathbb{R})}\\&=\partial_t g_{\pm}\big(t,X_t(x,v)\big)+\Big\langle\mathbf{v}\big(t,X_t(x,v)\big)\,,\nabla_{x,v}\,g_{\pm}\big(t,X_t(x,v)\big)\Big\rangle_{T_{(x,v)}(\mathbb{T}\times\mathbb{R})}.\end{aligned}$$

Now, we consider a parametrization $z_{\pm}(t,\cdot):\mathbb{T}\to\mathbb{T}\times\mathbb{R}$ of the boundary $\Gamma_t^{\pm}$, then

$$\partial_t g_{\pm}\big(t,z_{\pm}(t,x)\big)+\Big\langle\partial_t z_{\pm}(t,x)\,,\nabla_{x,v}\,g_{\pm}\big(t,z_{\pm}(t,x)\big)\Big\rangle_{T_{z_{\pm}(t,x)}(\mathbb{T}\times\mathbb{R})}=0.$$

In particular,

$$\Big\langle\partial_t z_{\pm}(t,x)-\mathbf{v}\big(t,z_{\pm}(t,x)\big)\,,\nabla_{x,v}\,g_{\pm}\big(t,z_{\pm}(t,x)\big)\Big\rangle_{T_{z_{\pm}(t,x)}(\mathbb{T}\times\mathbb{R})}=0.\tag{1.11}$$

Now, by construction, the vectors $\nabla_{x,v}\,g_{\pm}\big(t,z_{\pm}(t,x)\big)$ and $\partial_x z_{\pm}(t,x)$ are respectively orthogonal and transversal to $\Gamma_t^{\pm}$ inside $T_{z_{\pm}(t,x)}(\mathbb{T}\times\mathbb{R})$. Hence, we can write

$$\nabla_{x,v}\,g_{\pm}\big(t,z_{\pm}(t,x)\big)=\alpha\mathtt{J}_{x,v}\partial_x z_{\pm}(t,x),\qquad\alpha\in\mathbb{R}.$$

As a consequence, the identity (1.11) becomes

$$\Big\langle\partial_t z_{\pm}(t,x)\,,\mathtt{J}_{x,v}\partial_x z_{\pm}(t,x)\Big\rangle_{T_{z_{\pm}(t,x)}(\mathbb{T}\times\mathbb{R})}=\Big\langle\mathbf{v}\big(t,z_{\pm}(t,x)\big)\,,\mathtt{J}_{x,v}\partial_x z_{\pm}(t,x)\Big\rangle_{T_{z_{\pm}(t,x)}(\mathbb{T}\times\mathbb{R})}.\tag{1.12}$$

Therefore, using (1.6) and the fact that $\mathtt{J}_{x,v}$ is orthogonal for the scalar product on the tangent plane, we obtain

$$\begin{aligned}\partial_x\Big(\mathbf{\Psi}\big(t,z_{\pm}(t,x)\big)\Big)&=\Big\langle\nabla_{x,v}\mathbf{\Psi}\big(t,z_{\pm}(t,x)\big)\,,\partial_x z_{\pm}(t,x)\Big\rangle_{T_{z_{\pm}(t,x)}(\mathbb{T}\times\mathbb{R})}\\&=\Big\langle\mathtt{J}_{x,v}\nabla_{x,v}\mathbf{\Psi}\big(t,z_{\pm}(t,x)\big)\,,\mathtt{J}_{x,v}\partial_x z_{\pm}(t,x)\Big\rangle_{T_{z_{\pm}(t,x)}(\mathbb{T}\times\mathbb{R})}\\&=\Big\langle\nabla_{x,v}^{\perp}\mathbf{\Psi}\big(t,z_{\pm}(t,x)\big)\,,\mathtt{J}_{x,v}\partial_x z_{\pm}(t,x)\Big\rangle_{T_{z_{\pm}(t,x)}(\mathbb{T}\times\mathbb{R})}\\&=-\Big\langle\mathbf{v}\big(t,z_{\pm}(t,x)\big)\,,\mathtt{J}_{x,v}\partial_x z_{\pm}(t,x)\Big\rangle_{T_{z_{\pm}(t,x)}(\mathbb{T}\times\mathbb{R})}.\end{aligned}\tag{1.13}$$

Combining (1.12) and (1.13), we deduce the following equations

$$\Big\langle\partial_t z_{\pm}(t,x)\,,\mathtt{J}_{x,v}\partial_x z_{\pm}(t,x)\Big\rangle_{T_{z_{\pm}(t,x)}(\mathbb{T}\times\mathbb{R})}=-\partial_x\Big(\mathbf{\Psi}\big(t,z_{\pm}(t,x)\big)\Big).\tag{1.14}$$

We fix $(a,b)\in\mathbb{R}^2$ with $a<b$. We consider an initial domain S_0 close to the flat strip $S_{\text{flat}}(a,b)$ defined in (1.9) and with the same area

$$|S_0|=|S_{\text{flat}}(a,b)|=b-a.$$

We denote

$$f(t,x,v) = \frac{1}{b-a}\mathbf{1}_{S_t}(x,v), \qquad S_t = \big\{(x,v)\in\mathbb{T}\times\mathbb{R} \quad \text{s.t.} \quad v_-(t,x) < v < v_+(t,x)\big\} \tag{1.15}$$

the corresponding electron layer weak solution of (1.1). The area condition (1.8) writes in this context

$$\int_0^1 \big[v_+(t,x) - v_-(t,x)\big]dx = b-a. \tag{1.16}$$

We take as an ansatz

$$\begin{cases} z_+(t,x) = \big(x, b + r_+(t,x)\big), & \text{i.e.} \quad v_+(t,x) = b + r_+(t,x), \\ z_-(t,x) = \big(x, a + r_-(t,x)\big), & \text{i.e.} \quad v_-(t,x) = a + r_-(t,x). \end{cases} \tag{1.17}$$

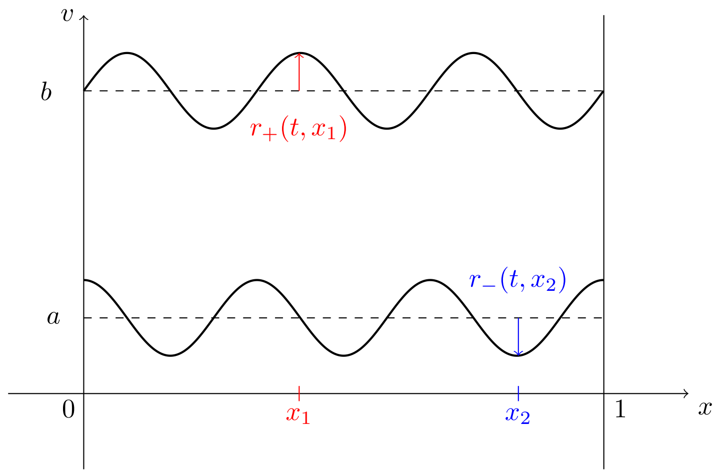

Figure 1: Perturbation of the flat strip $S_{\text{flat}}(a,b)$.

Observe that the area conservation condition (1.16) writes

$$\int_0^1 \big[r_+(t,x) - r_-(t,x)\big]dx = 0. \tag{1.18}$$

Now, on one hand

$$\begin{aligned} \Big\langle \partial_t z_\pm(t,x)\,,\, \mathtt{J}_{x,v}\partial_x z_\pm(t,x)\Big\rangle_{T_{z_\pm(t,x)}(\mathbb{T}\times\mathbb{R})} &= \Big\langle \partial_t r_\pm(t,x)\mathtt{e}_v\,,\, \mathtt{J}_{x,v}\big(\mathtt{e}_x + \partial_x r_\pm \mathtt{e}_v\big)\Big\rangle_{T_{z_\pm(t,x)}(\mathbb{T}\times\mathbb{R})} \\ &= \Big\langle \partial_t r_\pm(t,x)\mathtt{e}_v\,,\, \mathtt{e}_v - \partial_x r_\pm \mathtt{e}_x\Big\rangle_{T_{z_\pm(t,x)}(\mathbb{T}\times\mathbb{R})} \\ &= \partial_t r_\pm(t,x). \end{aligned} \tag{1.19}$$

On the other hand, using (1.6),

$$\begin{cases} \boldsymbol{\Psi}\big(t, z_+(t,x)\big) = \tfrac{1}{2}\big(b + r_+(t,x)\big)^2 + \boldsymbol{\varphi}(t,x), \\ \boldsymbol{\Psi}\big(t, z_-(t,x)\big) = \tfrac{1}{2}\big(a + r_-(t,x)\big)^2 + \boldsymbol{\varphi}(t,x). \end{cases} \tag{1.20}$$

In addition, from (1.4) and (1.15), we can write

$$\begin{aligned} \boldsymbol{\varphi}(t,x) &= \partial_{xx}^{-1}\left(1 - \int_{v_-(t,x)}^{v_+(t,x)} \frac{1}{b-a}dv\right) \\ &= \partial_{xx}^{-1}\Big(1 - \tfrac{1}{b-a}\big(v_+(t,x) - v_-(t,x)\big)\Big) \\ &= -\tfrac{1}{b-a}\partial_{xx}^{-1}\big(r_+(t,x) - r_-(t,x)\big). \end{aligned} \tag{1.21}$$

Inserting (1.19), (1.20) and (1.21) into (1.14), we end up with the following system

$$\begin{cases} \partial_t r_+(t,x) = -\partial_x\Big(\tfrac{1}{2}\big(b + r_+(t,x)\big)^2 - \tfrac{1}{b-a}\partial_{xx}^{-1}r_+(t,x) + \tfrac{1}{b-a}\partial_{xx}^{-1}r_-(t,x)\Big), \\ \partial_t r_-(t,x) = -\partial_x\Big(\tfrac{1}{2}\big(a + r_-(t,x)\big)^2 - \tfrac{1}{b-a}\partial_{xx}^{-1}r_+(t,x) + \tfrac{1}{b-a}\partial_{xx}^{-1}r_-(t,x)\Big), \end{cases} \tag{1.22}$$

which can recast in the following form

$$\begin{cases}\partial_t r_+(t,x)+\big(r_+(t,x)+b\big)\partial_x r_+(t,x)-\frac{1}{b-a}\partial_x^{-1}r_+(t,x)+\frac{1}{b-a}\partial_x^{-1}r_-(t,x)=0,\\ \partial_t r_-(t,x)+\big(r_-(t,x)+a\big)\partial_x r_-(t,x)-\frac{1}{b-a}\partial_x^{-1}r_+(t,x)+\frac{1}{b-a}\partial_x^{-1}r_-(t,x)=0,\end{cases}\tag{1.23}$$

where

$$\partial_x^{-1}\mathbf{e}_j\triangleq\frac{\mathbf{e}_j}{2\pi \mathrm{i}j},$$

or equivalently, in real notations,

$$\forall j\in\mathbb{N}^*,\qquad \partial_x^{-1}\cos(2\pi jx)\triangleq\frac{\sin(2\pi jx)}{2\pi j},\qquad \partial_x^{-1}\sin(2\pi jx)\triangleq-\frac{\cos(2\pi jx)}{2\pi j}\cdot\tag{1.24}$$

This is a system of two coupled quasilinear transport equations where the coupling is linear. Notice that if one tries to impose the constraint $r_+=r_-$, compatible with (1.18), in order tp reduce the study to a scalar equation, then one finds only the trivial solution $r_+=r_-=\mathrm{Cte}$, i.e. the flat strip, as a solution. Remark that (1.22) implies

$$\partial_t\int_0^1 r_+(t,x)dx=\partial_t\int_0^1 r_-(t,x)dx=0.$$

Then, we can impose

$$\int_0^1 r_+(t,x)dx=\int_0^1 r_-(t,x)dx=0,\tag{1.25}$$

which is compatible with (1.18).

1.3 Main results

Here, we expose our main results. Our inspiration naturally comes from the fluid mechanics where uniformly rotating vortex patch solutions were obtained for various models, see [5, 6, 12, 15, 16, 17, 20, 21, 22, 23, 25, 26, 29, 35, 36, 37]. In the planar case, such solutions are called *V-states* (for "Vortex states") according to the terminology introduced by Deem and Zabusky [11]. The second inspiration is borrowed (for instance) to the bifurcation of traveling waves for water-waves [7, 31, 34, 39, 40]. In honor of Deem and Zabusky terminology, we give the following definition.

Definition 1.1. ***(E-states or Electron-states)*** *Let $c\in\mathbb{R}$ and $\mathbf{m}\in\mathbb{N}^*$. We say that an electron layer solution to* (1.1) *is*

- $\mathbf{m}$*-symmetric if*

$$v_\pm\big(t,x+\tfrac{1}{\mathbf{m}}\big)=v_\pm(t,x),\qquad \text{i.e.}\qquad r_\pm\big(t,x+\tfrac{1}{\mathbf{m}}\big)=r_\pm(t,x).$$

- *an E(lectron)-state with velocity c if there exist 1-periodic profiles $\check{v}_\pm$, i.e. $\check{r}_\pm$, such that*

$$v_\pm(t,x)=\check{v}_\pm(x-ct),\qquad \text{i.e.}\qquad r_\pm(t,x)=\check{r}_\pm(x-ct).$$

The $\mathbf{m}$-symmetry condition for an E-state becomes

$$\check{v}_\pm\big(x+\tfrac{1}{\mathbf{m}}\big)=\check{v}_\pm(x),\qquad \text{i.e.}\qquad \check{r}_\pm\big(x+\tfrac{1}{\mathbf{m}}\big)=\check{r}_\pm(x).$$

Observe that the E-states are traveling periodic solutions for which the boundaries look stationary in a moving frame in translation with speed c in the x-direction. In what follows, we prove the emergence of E-states with analytic boundary. In particular, they do not present folding formation so there is no interaction with Dziurzynski's results. Moreover, for an electron-state, the electric field is time periodic. Since we work with non-smooth solutions close to non-smooth equilibria (patches), there is also no contradiction with the classical theory of Landau damping [14, 33]. We shall now state our main theorem.

Theorem 1.1. ***(Local bifurcation of E-states)***
The one dimensonal Vlasov-Poisson equation (1.1) *admits the following implicit solutions.*

(i) Let $a<b$ and $\mathbf{m}\in\mathbb{N}^*$. There exist two local curves

$$\mathscr{C}^{\pm,\mathbf{m}}_{\text{local}}(a,b)\triangleq\Big\{\big(c^{\pm}_{\mathbf{m}}(\mathbf{s},a,b),\check{r}^{\pm}_{\mathbf{m}}(\mathbf{s},a,b)\big),\quad |\mathbf{s}|<\delta\Big\},\qquad \delta>0$$

corresponding to $\mathbf{m}$-symmetric E-states bifurcating from the flat strip $S_{\text{flat}}(a,b)$ defined in (1.9) *and admitting the expansion*

$$c^{\pm}_{\mathbf{m}}(\mathbf{s},a,b)\underset{\mathbf{s}\to0}{=}c^{\pm}_{\mathbf{m}}(a,b)+\mathbf{s}^2c^{\pm}_{\mathbf{m},2}(a,b)+O(\mathbf{s}^3),$$

with

$$c_{\mathbf{m}}^{\pm}(a,b)\triangleq\frac{a+b}{2}\pm\sqrt{\frac{\pi^2\mathbf{m}^2(b-a)^2+1}{4\pi^2\mathbf{m}^2}},\qquad c_{\mathbf{m},2}^{+}(a,b)>0,\qquad c_{\mathbf{m},2}^{-}(a,b)<0$$

and

$$\check{r}_{\mathbf{m}}^{\pm}(\mathtt{s},a,b)(x)\underset{\mathtt{s}\to0}{=}\mathtt{s}\begin{pmatrix}2\pi\mathbf{m}\big(a-c_{\mathbf{m}}^{\pm}(a,b)\big)-\frac{1}{2\pi\mathbf{m}(b-a)}\\ -\frac{1}{2\pi\mathbf{m}(b-a)}\end{pmatrix}\cos(2\pi\mathbf{m}x)+O(\mathtt{s}^2).$$

Both bifurcations are of pitchfork-type and the bifurcation diagram admits (locally close to the trivial line) a "hyperbolic" structure as represented in the following figure

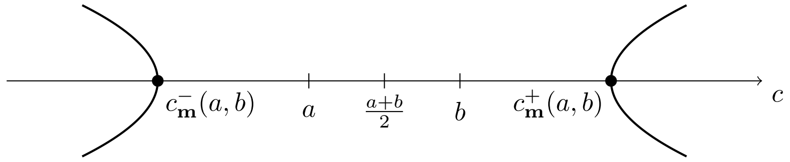

Figure 2: Representation of the velocity bifurcation diagram with "hyperbolic" structure.

(ii) *Let* $(a,c)\in\mathbb{R}^2$. *We denote, for any* $p\in\mathbb{R}^*$,

$$N_1(p)\triangleq 1+N_2(p),\qquad N_2(p)\triangleq\left\lfloor\tfrac{1}{2\pi|p|}\right\rfloor.$$

(a) *Assume* $a<c$. *Then, for any* $\mathbf{m}\in\mathbb{N}^*$ *with* $\mathbf{m}\geqslant N_1(c-a)$, *there exists a local curve*

$$\mathscr{C}_{\text{local}}^{\mathbf{m}}(a,c)\triangleq\Big\{\big(b_{\mathbf{m}}(\mathtt{s},a,c),\check{r}_{\mathbf{m}}(\mathtt{s},a,c)\big),\quad|\mathtt{s}|<\delta\Big\},\qquad\delta>0$$

corresponding to $\mathbf{m}$*-symmetric E-states with velocity* c *bifurcating from the flat strip* $S_{\text{flat}}\big(a,b_{\mathbf{m}}(a,c)\big)$ *and admitting the following expansion*

$$b_{\mathbf{m}}(\mathtt{s},a,c)\underset{\mathtt{s}\to0}{=}b_{\mathbf{m}}(a,c)+\mathtt{s}^2b_{\mathbf{m},2}(a,c)+O(\mathtt{s}^3),$$

with

$$b_{\mathbf{m}}(a,c)\triangleq c+\frac{1}{4\pi^2\mathbf{m}^2(a-c)},\qquad b_{\mathbf{m},2}(a,c)<0\ \text{(subcritical bifurcation)}$$

and

$$\check{r}_{\mathbf{m}}(\mathtt{s},a,c)(x)\underset{\mathtt{s}\to0}{=}\mathtt{s}\begin{pmatrix}2\pi\mathbf{m}(a-c)-\frac{1}{2\pi\mathbf{m}\big(b_{\mathbf{m}}(a,c)-a\big)}\\ -\frac{1}{2\pi\mathbf{m}\big(b_{\mathbf{m}}(a,c)-a\big)}\end{pmatrix}\cos(2\pi\mathbf{m}x)+O(\mathtt{s}^2).$$

(b) *Assume* $c<a<c+\frac{1}{2\pi}$. *Then, for any* $\mathbf{m}\in[\![1,N_2(a-c)]\!]$, *there exists a local curve* $\mathscr{C}_{\text{local}}^{\mathbf{m}}(a,c)$ *of* $\mathbf{m}$*-symmetric E-states with velocity* c *bifucating from the flat strip* $S_{\text{flat}}\big(a,b_{\mathbf{m}}(a,c)\big)$ *as above but with* $b_{\mathbf{m},2}(a,c)>0$ *(supercritical bifurcation).*

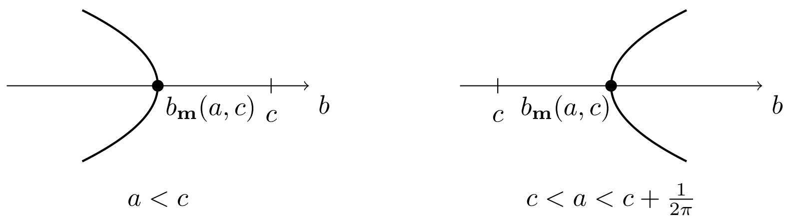

Figure 3: Representation of b-local bifurcation diagram close to asymmetric flat strips.

(iii) *Let* $(b,c)\in\mathbb{R}^2$.

(a) *Assume* $b>c$. *Then, for any* $\mathbf{m}\in\mathbb{N}^*$ *with* $\mathbf{m}\geqslant N_1(b-c)$, *there exists a local curve*

$$\mathscr{C}_{\text{local}}^{\mathbf{m}}(b,c)\triangleq\Big\{\big(a_{\mathbf{m}}(\mathtt{s},b,c),\check{r}_{\mathbf{m}}(\mathtt{s},b,c)\big),\quad|\mathtt{s}|<\delta\Big\},\qquad\delta>0$$

corresponding to $\mathbf{m}$*-symmetric E-states with velocity* c *bifurcating from the flat strip* $S_{\text{flat}}\big(a_{\mathbf{m}}(b,c),b\big)$ *and admitting the following expansion*

$$a_{\mathbf{m}}(\mathtt{s},b,c) \underset{\mathtt{s}\to 0}{=} a_{\mathbf{m}}(b,c)+\mathtt{s}^2 a_{\mathbf{m},2}(b,c)+O(\mathtt{s}^3),$$

with

$$a_{\mathbf{m}}(b,c)\triangleq c+\frac{1}{4\pi^2\mathbf{m}^2(b-c)},\qquad a_{\mathbf{m},2}(b,c)>0 \text{ (supercritical bifurcation)}$$

and

$$\check{r}_{\mathbf{m}}(\mathtt{s},b,c)(x) \underset{\mathtt{s}\to 0}{=} \mathtt{s}\begin{pmatrix} 2\pi\mathbf{m}\big(a_{\mathbf{m}}(b,c)-c\big)-\frac{1}{2\pi\mathbf{m}\big(b-a_{\mathbf{m}}(b,c)\big)} \\ -\frac{1}{2\pi\mathbf{m}\big(b-a_{\mathbf{m}}(b,c)\big)}\end{pmatrix}\cos(2\pi\mathbf{m}x)+O(\mathtt{s}^2).$$

(b) *Assume* $c-\frac{1}{2\pi}<b<c$*. Then, for any* $\mathbf{m}\in[\![1,N_2(c-b)]\!]$*, there exists a local curve* $\mathscr{C}^{\mathbf{m}}_{\text{local}}(b,c)$ *of* $\mathbf{m}$*-symmetric E-states with velocity* c *bifurcating from the flat strip* $S_{\text{flat}}\big(a_{\mathbf{m}}(b,c),b\big)$ *as above but with* $a_{\mathbf{m},2}(b,c)<0$ *(subcritical bifurcation).*

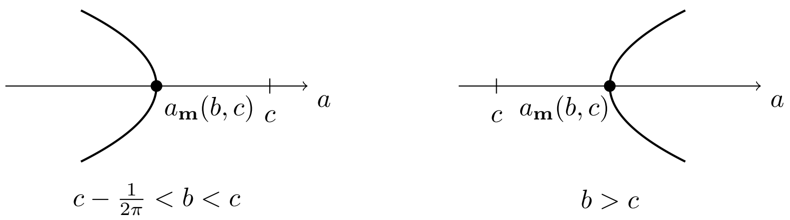

Figure 4: Representation of a-local bifurcation diagram close to asymmetric flat strips.

(iv) *Let* $c\in\mathbb{R}^*$*. Then, for any* $\mathbf{m}\in\mathbb{N}^*$ *with* $\mathbf{m}\geqslant N_1(c)$*, there exists a local curve*

$$\mathscr{C}^{\mathbf{m}}_{\text{local}}(c)\triangleq\Big\{\big(a_{\mathbf{m}}(\mathtt{s},c),\check{r}_{\mathbf{m}}(\mathtt{s},c)\big),\quad |\mathtt{s}|<\delta\Big\},\qquad \delta>0$$

corresponding to $\mathbf{m}$*-symmetric E-states with velocity* c *bifurcating from the symmetric flat strip* $S_{\text{flat}}\big(-a_{\mathbf{m}}(c),a_{\mathbf{m}}(c)\big)$ *and admitting the following expansion*

$$a_{\mathbf{m}}(\mathtt{s},c) \underset{\mathtt{s}\to 0}{=} a_{\mathbf{m}}(c)+\mathtt{s}^2 a_{\mathbf{m},2}(c)+O(\mathtt{s}^3),$$

with

$$a_{\mathbf{m}}(c)\triangleq\sqrt{\frac{4\pi^2\mathbf{m}^2c^2-1}{4\pi^2\mathbf{m}^2}},\qquad \begin{cases} a_{\mathbf{m},2}(c)<0, & \text{if } c>0,\\ a_{\mathbf{m},2}(c)>0, & \text{if } c<0\end{cases}$$

and

$$\check{r}_{\mathbf{m}}(\mathtt{s},c)(x) \underset{\mathtt{s}\to 0}{=} \mathtt{s}\begin{pmatrix} 2\pi\mathbf{m}\big(a_{\mathbf{m}}(c)-c\big)-\frac{1}{4\pi\mathbf{m}a_{\mathbf{m}}(c)} \\ -\frac{1}{4\pi\mathbf{m}a_{\mathbf{m}}(c)}\end{pmatrix}\cos(2\pi\mathbf{m}x)+O(\mathtt{s}^2).$$

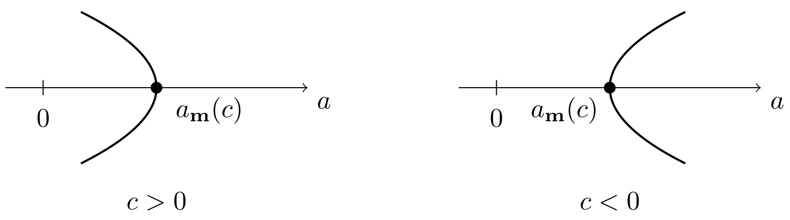

Figure 5: Representation of the area local bifurcation diagram close to symmetric flat strips.

In addition, each one of the previous bifurcations occurs at any level of Sobolev-analytic regularity $H^{s,\sigma}$ *for* $\sigma>0$ *and* $s\geqslant 1$*.*

To prove Theorem 1.1 we use Crandall-Rabinowitz-Shi's Theorem A.1 for the construction of the local curves and study the pitchfork phenomenon. Then, we apply Buffoni-Toland Theorem A.2 to globally extend the branches and get the following result.

Theorem 1.2. ***(Global bifurcation of E-states)*** *All the bifurcations of Theorem 1.1 are global in $H^{s,\sigma}$ for $s>\frac{3}{2}$ and $\sigma>0$. More precisely,*

(i) Let $a<b$ and $\mathbf{m}\in\mathbb{N}^*$. Fix $\kappa\in\{+,-\}$. Then, there exist two global curves

$$\mathscr{C}_{\text{global}}^{\kappa,\mathbf{m}}(a,b)\triangleq\Big\{\big(c_{\mathbf{m}}^{\kappa}(\mathtt{s},a,b),\check{r}_{\mathbf{m}}^{\kappa}(\mathtt{s},a,b)\big),\quad \mathtt{s}\in\mathbb{R}\Big\}$$

corresponding to $\mathbf{m}$-symmetric E-states and extending the local curves $\mathscr{C}_{\text{local}}^{\kappa,\mathbf{m}}(a,b)$ given by Theorem 1.1-(i). Moreover, the curve $\mathscr{C}_{\text{global}}^{\kappa,\mathbf{m}}(a,b)$ admits locally around each of its points a real-analytic reparametrization. In addition, one has the following alternatives

(A1) There exist $T_{\mathbf{m}}^{\kappa}(a,b)>0$ such that

$$\forall\mathtt{s}\in\mathbb{R},\quad c_{\mathbf{m}}^{\kappa}\big(\mathtt{s}+T_{\mathbf{m}}^{\kappa}(a,b),a,b\big)=c_{\mathbf{m}}^{\kappa}(\mathtt{s},a,b)\qquad\text{and}\qquad \check{r}_{\mathbf{m}}^{\kappa}\big(\mathtt{s}+T_{\mathbf{m}}^{\kappa}(a,b),a,b\big)=\check{r}_{\mathbf{m}}^{\kappa}(\mathtt{s},a,b).$$

(A2) One of the following limits occurs (possibly simultaneously)

- *(Blow-up)* $\displaystyle\lim_{\mathtt{s}\to\pm\infty}\frac{1}{1+|c_{\mathbf{m}}^{\kappa}(\mathtt{s},a,b)|+\|\check{r}_{\mathbf{m}}^{\kappa}(\mathtt{s},a,b)\|_{s,\sigma}}=0.$
- *(Collision of the boundaries)* $\displaystyle\lim_{\mathtt{s}\to\pm\infty}\min_{x\in\mathbb{T}}\Big|\big(\check{r}_{\mathbf{m}}^{\kappa}\big)_{+}(\mathtt{s},a,b)(x)-\big(\check{r}_{\mathbf{m}}^{\kappa}\big)_{-}(\mathtt{s},a,b)(x)+b-a\Big|=0.$
- *(Degeneracy +)* $\displaystyle\lim_{\mathtt{s}\to\pm\infty}\min_{x\in\mathbb{T}}\Big|\big(\check{r}_{\mathbf{m}}^{\kappa}\big)_{+}(\mathtt{s},a,b)(x)+b-c_{\mathbf{m}}^{\pm}(\mathtt{s},a,b)\Big|=0.$
- *(Degeneracy −)* $\displaystyle\lim_{\mathtt{s}\to\pm\infty}\min_{x\in\mathbb{T}}\Big|\big(\check{r}_{\mathbf{m}}^{\kappa}\big)_{-}(\mathtt{s},a,b)(x)+a-c_{\mathbf{m}}^{\pm}(\mathtt{s},a,b)\Big|=0.$

(ii) Let $(a,c)\in\mathbb{R}^2$, $\mathbf{m}\in\mathbb{N}^*$ with $a<c$ and $\mathbf{m}\geqslant N_1(c-a)$ (resp. $c<a<c+\frac{1}{2\pi}$ and $\mathbf{m}\in[\![1,N_2(a-c)]\!]$). Then, there exists a global curve

$$\mathscr{C}_{\text{global}}^{\mathbf{m}}(a,c)\triangleq\Big\{\big(b_{\mathbf{m}}(\mathtt{s},a,c),\check{r}_{\mathbf{m}}(\mathtt{s},a,c)\big),\quad \mathtt{s}\in\mathbb{R}\Big\}$$

corresponding to $\mathbf{m}$-symmetric E-states and extending the local curves $\mathscr{C}_{\text{local}}^{\mathbf{m}}(a,c)$ given by Theorem 1.1-(ii). Moreover, the curve $\mathscr{C}_{\text{global}}^{\mathbf{m}}(a,c)$ admits locally around each of its points a real-analytic reparametrization. In addition, one has the following alternatives

(A1) There exist $T_{\mathbf{m}}(a,c)>0$ such that

$$\forall\mathtt{s}\in\mathbb{R},\quad b_{\mathbf{m}}\big(\mathtt{s}+T_{\mathbf{m}}(a,c),a,c\big)=b_{\mathbf{m}}(\mathtt{s},a,c)\qquad\text{and}\qquad \check{r}_{\mathbf{m}}\big(\mathtt{s}+T_{\mathbf{m}}(a,c),a,c\big)=\check{r}_{\mathbf{m}}(\mathtt{s},a,c).$$

(A2) One of the following limits occurs (possibly simultaneously)

- *(Blow-up)* $\displaystyle\lim_{\mathtt{s}\to\pm\infty}\frac{1}{1+|b_{\mathbf{m}}(\mathtt{s},a,c)|+\|\check{r}_{\mathbf{m}}(\mathtt{s},a,c)\|_{s,\sigma}}=0.$
- *(Collision of the boundaries)* $\displaystyle\lim_{\mathtt{s}\to\pm\infty}\min_{x\in\mathbb{T}}\Big|\big(\check{r}_{\mathbf{m}}\big)_{+}(\mathtt{s},a,c)(x)-\big(\check{r}_{\mathbf{m}}\big)_{-}(\mathtt{s},a,c)(x)+b_{\mathbf{m}}(\mathtt{s},a,c)-a\Big|=0.$
- *(Degeneracy +)* $\displaystyle\lim_{\mathtt{s}\to\pm\infty}\min_{x\in\mathbb{T}}\Big|\big(\check{r}_{\mathbf{m}}\big)_{+}(\mathtt{s},a,c)(x)+b_{\mathbf{m}}(\mathtt{s},a,c)-c\Big|=0.$
- *(Degeneracy −)* $\displaystyle\lim_{\mathtt{s}\to\pm\infty}\min_{x\in\mathbb{T}}\Big|\big(\check{r}_{\mathbf{m}}\big)_{-}(\mathtt{s},a,c)(x)+a-c\Big|=0.$

(iii) Let $(b,c)\in\mathbb{R}^2$, $\mathbf{m}\in\mathbb{N}^*$ with $b>c$ and $\mathbf{m}\geqslant N_1(b-c)$ (resp. $c-\frac{1}{2\pi}<b<c$ and $\mathbf{m}\in[\![1,N_2(c-b)]\!]$). Then, there exists a global curve

$$\mathscr{C}_{\text{global}}^{\mathbf{m}}(b,c)\triangleq\Big\{\big(a_{\mathbf{m}}(\mathtt{s},b,c),\check{r}_{\mathbf{m}}(\mathtt{s},b,c)\big),\quad \mathtt{s}\in\mathbb{R}\Big\}$$

corresponding to $\mathbf{m}$-symmetric E-states and extending the local curves $\mathscr{C}_{\text{local}}^{\mathbf{m}}(b,c)$ given by Theorem 1.1-(iii). Moreover, the curve $\mathscr{C}_{\text{global}}^{\mathbf{m}}(b,c)$ admits locally around each of its points a real-analytic reparametrization. In addition, one has the following alternatives

(A1) There exist $T_{\mathbf{m}}(b,c)>0$ such that

$$\forall\mathtt{s}\in\mathbb{R},\quad a_{\mathbf{m}}\big(\mathtt{s}+T_{\mathbf{m}}(b,c),b,c\big)=a_{\mathbf{m}}(\mathtt{s},b,c)\qquad\text{and}\qquad \check{r}_{\mathbf{m}}\big(\mathtt{s}+T_{\mathbf{m}}(b,c),b,c\big)=\check{r}_{\mathbf{m}}(\mathtt{s},b,c).$$

(A2) One of the following limits occurs (possibly simultaneously)

- *(Blow-up)* $\displaystyle\lim_{\mathtt{s}\to\pm\infty}\frac{1}{1+|a_{\mathbf{m}}(\mathtt{s},b,c)|+\|\check{r}_{\mathbf{m}}(\mathtt{s},b,c)\|_{s,\sigma}}=0.$

- *(Collision of the boundaries)* $\lim\limits_{\mathtt{s}\to\pm\infty}\min\limits_{x\in\mathbb{T}}\Big|\big(\check{r}_{\mathbf{m}}\big)_{+}(\mathtt{s},b,c)(x)-\big(\check{r}_{\mathbf{m}}\big)_{-}(\mathtt{s},b,c)(x)+b-a_{\mathbf{m}}(\mathtt{s},b,c)\Big|=0.$
- *(Degeneracy +)* $\lim\limits_{\mathtt{s}\to\pm\infty}\min\limits_{x\in\mathbb{T}}\Big|\big(\check{r}_{\mathbf{m}}\big)_{+}(\mathtt{s},b,c)(x)+b-c\Big|=0.$
- *(Degeneracy −)* $\lim\limits_{\mathtt{s}\to\pm\infty}\min\limits_{x\in\mathbb{T}}\Big|\big(\check{r}_{\mathbf{m}}\big)_{-}(\mathtt{s},b,c)(x)+a_{\mathbf{m}}(\mathtt{s},b,c)-c\Big|=0.$

(iv) Let $c\in\mathbb{R}^*$ and $\mathbf{m}\in\mathbb{N}^*$ with $\mathbf{m}\geqslant N_1(c)$. Then, there exists a global curve

$$\mathscr{C}^{\mathbf{m}}_{\textnormal{global}}(c)\triangleq\Big\{\big(a_{\mathbf{m}}(\mathtt{s},c),\check{r}_{\mathbf{m}}(\mathtt{s},c)\big),\quad \mathtt{s}\in\mathbb{R}\Big\}$$

corresponding to $\mathbf{m}$-symmetric E-states and extending the local curves $\mathscr{C}^{\mathbf{m}}_{\textnormal{local}}(c)$ given by Theorem 1.1-(iv). Moreover, the curve $\mathscr{C}^{\mathbf{m}}_{\textnormal{global}}(c)$ admits locally around each of its points a real-analytic reparametrization. In addition, one has the following alternatives

(A1) There exist $T_{\mathbf{m}}(c)>0$ such that

$$\forall\mathtt{s}\in\mathbb{R},\quad a_{\mathbf{m}}\big(\mathtt{s}+T_{\mathbf{m}}(c),c\big)=a_{\mathbf{m}}(\mathtt{s},c)\qquad\text{and}\qquad \check{r}_{\mathbf{m}}\big(\mathtt{s}+T_{\mathbf{m}}(c),c\big)=\check{r}_{\mathbf{m}}(\mathtt{s},c).$$

(A2) One of the following limits occurs (possibly simultaneously)

- *(Blow-up)* $\lim\limits_{\mathtt{s}\to\pm\infty}\dfrac{1}{1+a_{\mathbf{m}}(\mathtt{s},c)+\|\check{r}_{\mathbf{m}}(\mathtt{s},c)\|_{s,\sigma}}=0.$
- *(Collision of the boundaries)* $\lim\limits_{\mathtt{s}\to\pm\infty}\min\limits_{x\in\mathbb{T}}\Big|\big(\check{r}_{\mathbf{m}}\big)_{+}(\mathtt{s},c)(x)-\big(\check{r}_{\mathbf{m}}\big)_{-}(\mathtt{s},c)(x)+2a_{\mathbf{m}}(\mathtt{s},c)\Big|=0.$
- *(Degeneracy +)* $\lim\limits_{\mathtt{s}\to\pm\infty}\min\limits_{x\in\mathbb{T}}\Big|\big(\check{r}_{\mathbf{m}}\big)_{+}(\mathtt{s},c)(x)+a_{\mathbf{m}}(\mathtt{s},c)-c\Big|=0.$
- *(Degeneracy −)* $\lim\limits_{\mathtt{s}\to\pm\infty}\min\limits_{x\in\mathbb{T}}\Big|\big(\check{r}_{\mathbf{m}}\big)_{-}(\mathtt{s},c)(x)-a_{\mathbf{m}}(\mathtt{s},c)-c\Big|=0.$

Remark 1.2. *We first make the following remarks concerning Theorem 1.1.*

1. *The "hyperbolic" structure of the bifurcation diagram (Figure 2) is in contrast with the Eulerian case for doubly-connected patches [24, 28] where we have an "elliptic" situation.*
2. *The fact of having a finite number of bifurcation points in the asymmetric case is also very interesting. This rarely happens in the fluid patch class, see [18, Thm. 1.1-(i)].*
3. *A precise expression of $c^{\pm}_{\mathbf{m},2}(a,b)$, $b_{\mathbf{m},2}(a,c)$, $a_{\mathbf{m},2}(b,c)$ and $a_{\mathbf{m},2}(c)$ will be given along Section 2. More generally, one can obtain a general expansion of the solutions by an algorithmic procedure inserting an a priori unknown expansion into the equations and solving the resulting system order by order with respect to the parametrization parameter $\mathtt{s}$.*
4. *For the items (ii) and (iii), the case $c=0$ (corresponding to stationary solutions) can be reached for suitable ranges of a or b. This makes echo to [19].*

Now, let us discuss the conclusions of Theorem 1.2.

1. *Conversely to [28] and similarly to [23], we expect that the loop alternative (A1) does not occur. This may require a more refined analysis introducing suitable nodal conditions and reformulating the study as a Riemann-Hilbert problem.*
2. *As we shall see in Section 3.1, for a true solution of the system (2.1) (corresponding to E-states), the "Degeneracy ±" alternative happens for a critical point of $\check{r}_{\mp}$. Hence, for the limiting E-states (end of the branch), we expect the formation of corners (as conjectured in the case of V-states).*

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2 Local construction

This section is devoted to the local construction of the branches. For that purpose, we reformulate the problem by looking for the zeros of a nonlinear time independant functional. The flat strips correspond to a trivial line of roots when only one of the free parameters varies. Then, we implement in a systematic way the Crandall-Rabinowitz Theorem A.1 by looking at the kernel, range and transversality conditions. We also investigate the pitchfork bifurcation property of the constructed branches by computing the condition given by Shi in [38], see also Theorem A.1.

We shall look for solutions of (1.23) in the form

$$r_{\pm}(t,x)=\check{r}_{\pm}(x-ct),\qquad c\in\mathbb{R},\qquad \check{r}_{\pm}\in L^2(\mathbb{T}).$$

With this ansatz, the system (1.23) becomes

$$\begin{aligned}&\forall x\in\mathbb{T},\quad F(a,b,c,\check{r}_+,\check{r}_-)(x)=0,\qquad F\triangleq(F_+\,,\,F_-),\\&F_+(a,b,c,\check{r}_+,\check{r}_-)(x)\triangleq\big(\check{r}_+(x)+b-c\big)\partial_x\check{r}_+(x)-\tfrac{1}{b-a}\partial_x^{-1}\check{r}_+(x)+\tfrac{1}{b-a}\partial_x^{-1}\check{r}_-(x),\\&F_-(a,b,c,\check{r}_+,\check{r}_-)(x)\triangleq\big(\check{r}_-(x)+a-c\big)\partial_x\check{r}_-(x)-\tfrac{1}{b-a}\partial_x^{-1}\check{r}_+(x)+\tfrac{1}{b-a}\partial_x^{-1}\check{r}_-(x).\end{aligned}\tag{2.1}$$

One readily has

$$\forall a<b,\quad\forall c\in\mathbb{R},\quad F(a,b,c,0,0)=0.\tag{2.2}$$

This identity provides, either for fixed couple (a,b), (a,c) or (b,c) a line of trivial solutions corresponding to the flat strip(s), see Lemma 1.1. We shall find non-trivial solutions of (2.1) by implementing the local bifurcation theory through the use of Crandall-Rabinowitz Theorem A.1. We shall work with the following Sobolev-analytic function spaces defined for any $s,\sigma\geqslant 0$ and $\mathbf{m}\in\mathbb{N}^*$ by

$$X^{s,\sigma}_{\mathbf{m}}\triangleq\Big\{f=(f_+,f_-),\qquad\forall x\in\mathbb{T},\quad f_{\pm}(x)=\sum_{j=1}^{\infty}f_j^{\pm}\cos(2\pi j\mathbf{m}x),\quad f_j^{\pm}\in\mathbb{R},\quad\sum_{j=1}^{\infty}|2\pi j|^{2s}|f_j^{\pm}|^2e^{4\pi\sigma|j|}<\infty\Big\},$$

$$Y^{s,\sigma}_{\mathbf{m}}\triangleq\Big\{g=(g_+,g_-),\qquad\forall x\in\mathbb{T},\quad g_{\pm}(x)=\sum_{j=1}^{\infty}g_j^{\pm}\sin(2\pi j\mathbf{m}x),\quad g_j^{\pm}\in\mathbb{R},\quad\sum_{j=1}^{\infty}|2\pi j|^{2s}|g_j^{\pm}|^2e^{4\pi\sigma|j|}<\infty\Big\}.$$

Both spaces are endowed with the norm

$$\|(u_+,u_-)\|_{s,\sigma}\triangleq\|u_+\|_{s,\sigma}+\|u_-\|_{s,\sigma},\qquad\|u_{\pm}\|_{s,\sigma}\triangleq\left(\sum_{j=1}^{\infty}(2\pi j)^{2s}(u_j^{\pm})^2e^{4\pi\sigma j}\right)^{\frac12}.$$

The average is taken equal to zero in accordance with (1.25). We refer the reader for instance to [1] for a nice introduction to the general Sobolev-analytic spaces and there properties. We mention that the Sobolev-analytic scale $(H^{s,\sigma})_{s\geqslant0,\sigma\geqslant0}$ behaves like the classical Sobolev scale $(H^s)_{s\geqslant0}$ and admits the same properties with respect to the Sobolev regularity parameter s (interpolation, product and composition laws, compact embeddings etc...). One can easily check from the structure (2.1), using in particular the classical formula

$$\forall(u,v)\in\mathbb{R}^2,\quad\sin(u)\cos(v)=\tfrac12\big(\sin(u+v)+\sin(u-v)\big),\tag{2.3}$$

that, denoting $\mathbb{S}\triangleq\{(x,y)\in\mathbb{R}^2\quad\text{s.t.}\quad x<y\}$,

$$\text{for any }\sigma>0\text{ and }s\geqslant1,\ \text{the function }F:\mathbb{S}\times\mathbb{R}\times X^{s,\sigma}_{\mathbf{m}}\to Y^{s-1,\sigma}_{\mathbf{m}}\text{ is well-defined and analytic.}\tag{2.4}$$

Indeed, the analyticity results for that (2.1) only involves linear or quadratic terms. From now on, we fix $s\geqslant1$ and $\sigma>0$. The linearized operator at $(\check{r}_+,\check{r}_-)=(0,0)$ in the direction (h_+,h_-) is

$$d_{(\check{r}_+,\check{r}_-)}F(a,b,c,0,0)[h_+,h_-]=I_{(0,0)}[h_+,h_-]+K_{(0,0)}[h_+,h_-],\tag{2.5}$$

with

$$I_{(0,0)}\triangleq\begin{pmatrix}(b-c)\partial_x&0\\0&(a-c)\partial_x\end{pmatrix},\qquad K_{(0,0)}\triangleq\tfrac{1}{b-a}\begin{pmatrix}-\partial_x^{-1}&\partial_x^{-1}\\-\partial_x^{-1}&\partial_x^{-1}\end{pmatrix}.\tag{2.6}$$

If $c\notin\{a,b\}$, then $I_{(0,0)}:X^{s,\sigma}_{\mathbf{m}}\to Y^{s-1,\sigma}_{\mathbf{m}}$ is an isomorphism. In addition, $K_{(0,0)}:X^{s,\sigma}_{\mathbf{m}}\to Y^{s+1,\sigma}_{\mathbf{m}}$ is continuous and by Rellich-type Theorem, we deduce that $K_{(0,0)}:X^{s,\sigma}_{\mathbf{m}}\to Y^{s-1,\sigma}_{\mathbf{m}}$ is a compact operator. Therefore,

$$\text{for }c\notin\{a,b\},\ \sigma>0\text{ and }s\geqslant1,\ d_{(\check{r}_+,\check{r}_-)}F(a,b,c,0,0):X^{s,\sigma}_{\mathbf{m}}\to Y^{s-1,\sigma}_{\mathbf{m}}\text{ is a zero index Fredholm operator.}\tag{2.7}$$

Moreover, this latter admits the following Fourier representation : for given $(h_+, h_-) \in X^{s,\sigma}_{\mathbf{m}}$ in the form

$$h_\pm(x) = \sum_{j=1}^{\infty} h_j^\pm \cos(2\pi j\mathbf{m}x), \qquad h_j^\pm \in \mathbb{R},$$

we have

$$d_{(\check{r}_+,\check{r}_-)}F(a,b,c,0,0)[h_+,h_-](x) = -\sum_{j=1}^{\infty} M_{j\mathbf{m}}(a,b,c)\begin{pmatrix} h_j^+ \\ h_j^- \end{pmatrix}\sin(2\pi j\mathbf{m}x), \tag{2.8}$$

where

$$M_j(a,b,c) \triangleq \begin{pmatrix} 2\pi j(b-c) + \frac{1}{2\pi j(b-a)} & -\frac{1}{2\pi j(b-a)} \\ \frac{1}{2\pi j(b-a)} & 2\pi j(a-c) - \frac{1}{2\pi j(b-a)} \end{pmatrix}. \tag{2.9}$$

In order to apply Crandall-Rabinowitz Theorem, we shall look for the singularity of the matrices $M_j(a,b,c)$ giving rise in turn to a non-trivial kernel for the linearized operator. For any $j \in \mathbb{N}^*$, the determinant of $M_j(a,b,c)$ is

$$\Delta_j(a,b,c) \triangleq \det\big(M_j(a,b,c)\big) = 4\pi^2 j^2 (b-c)(a-c) - 1. \tag{2.10}$$

We shall now dissociate the analysis whenever two of the three parameters a, b or c are fixed, which leads to the following subsections. We also discuss the symmetric case $(a,b) = (-a,a)$, with $a > 0$ which must be treated separately.

2.1 Velocity bifurcation

In this subsection, we fix $a < b$ and study the bifurcation with respect to the parameter $c \in \mathbb{R}$. First, we can rewrite the determinant $\Delta_j(a,b,c)$ in (2.10) as a polynomial of degree two in the variable c as follows

$$\Delta_j(a,b,c) = 4\pi^2 j^2 c^2 - 4\pi^2 j^2 (a+b)c + 4\pi^2 j^2 ab - 1. \tag{2.11}$$

The associated discriminant is

$$\begin{aligned} \delta_j(a,b) &\triangleq 16\pi^4 j^4 (a+b)^2 + 16\pi^2 j^2 (1 - 4\pi^2 j^2 ab) \\ &= 16\pi^2 j^2 \big(\pi^2 j^2 (b-a)^2 + 1\big) > 0. \end{aligned}$$

We deduce that

$$\Delta_j(a,b,c) = 0 \quad \Leftrightarrow \quad c = c_j^\pm(a,b) \triangleq \frac{a+b}{2} \pm \sqrt{\frac{\pi^2 j^2 (b-a)^2 + 1}{4\pi^2 j^2}}. \tag{2.12}$$

▸ **One dimensional kernel condition :** The sequence $\big(c_j^+(a,b)\big)_{j\in\mathbb{N}^*}$ is decreasing and tends to b when $j \to \infty$ whereas the sequence $\big(c_j^-(a,b)\big)_{j\in\mathbb{N}^*}$ is increasing and tends to a when $j \to \infty$. As a consequence, for any fixed $\mathbf{m} \in \mathbb{N}^*$,

$$\Delta_{\mathbf{m}}\big(a,b,c_{\mathbf{m}}^\pm(a,b)\big) = 0 \qquad \text{and} \qquad \forall j \in \mathbb{N}\setminus\{0,1\}, \quad \Delta_{j\mathbf{m}}\big(a,b,c_{\mathbf{m}}^\pm(a,b)\big) \neq 0.$$

Actually, from (2.10), we deduce

$$\begin{aligned} \forall j \in \mathbb{N}\setminus\{0,1\}, \quad \Delta_{j\mathbf{m}}\big(a,b,c_{\mathbf{m}}^\pm(a,b)\big) &= 4\pi^2 \mathbf{m}^2 j^2 \big(b - c_{\mathbf{m}}^\pm(a,b)\big)\big(a - c_{\mathbf{m}}^\pm(a,b)\big) - 1 \\ &= j^2 - 1 > 0. \end{aligned} \tag{2.13}$$

Thus, the kernel of $d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^\pm(a,b),0,0\big)$ is one dimensional, more precisely

$$\begin{aligned} &\ker\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^\pm(a,b),0,0\big)\Big) = \mathtt{span}\big(\check{r}_{0,\mathbf{m},a,b}^\pm\big), \\ &\check{r}_{0,\mathbf{m},a,b}^\pm(x) \triangleq \begin{pmatrix} 2\pi\mathbf{m}\big(a - c_{\mathbf{m}}^\pm(a,b)\big) - \frac{1}{2\pi\mathbf{m}(b-a)} \\ -\frac{1}{2\pi\mathbf{m}(b-a)} \end{pmatrix}\cos(2\pi\mathbf{m}x). \end{aligned} \tag{2.14}$$

▸ **Range condition :** Notice that the monotonicity and the convergence of the sequences $\big(c_j^\pm(a,b)\big)_{j\in\mathbb{N}^*}$ give that $c_{\mathbf{m}}^\pm(a,b) \notin \{a,b\}$. So the Fredholmness property (2.7) together with the previous point imply that the range $\mathcal{R}\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^\pm(a,b),0,0\big)\Big)$ is closed and of codimension one in $Y_{\mathbf{m}}^{s-1,\sigma}$. To check, later on, the transversality, we shall prove that

$$\begin{aligned} &\mathcal{R}\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^\pm(a,b),0,0\big)\Big) = V^\perp, \\ &V \triangleq \mathtt{span}\big(g_{0,\mathbf{m},a,b}^\pm\big), \qquad g_{0,\mathbf{m},a,b}^\pm(x) \triangleq \begin{pmatrix} 2\pi\mathbf{m}\big(a - c_{\mathbf{m}}^\pm(a,b)\big) - \frac{1}{2\pi\mathbf{m}(b-a)} \\ \frac{1}{2\pi\mathbf{m}(b-a)} \end{pmatrix}\sin(2\pi\mathbf{m}x), \end{aligned} \tag{2.15}$$

where the orthogonal is understood in the sense of the scalar product defined on $Y_{\mathbf{m}}^{s-1,\sigma}$ as follows: for any $g=(g_+,g_-)\in Y_{\mathbf{m}}^{s-1,\sigma}$ and $\widetilde{g}=(\widetilde{g}_+,\widetilde{g}_-)\in Y_{\mathbf{m}}^{s-1,\sigma}$ writing

$$g_\pm(x)=\sum_{j=1}^{\infty} g_j^\pm \sin(2\pi j\mathbf{m}x),\qquad \widetilde{g}_\pm(x)=\sum_{j=1}^{\infty}\widetilde{g}_j^\pm \sin(2\pi j\mathbf{m}x),\qquad g_j^\pm,\widetilde{g}_j^\pm\in\mathbb{R},\tag{2.16}$$

the scalar product $\langle g,\widetilde{g}\rangle$ of g and $\widetilde{g}$ is given by

$$\langle g,\widetilde{g}\rangle\triangleq\int_0^1\big(g_+(x)\widetilde{g}_+(x)+g_-(x)\widetilde{g}_-(x)\big)dx=\tfrac{1}{2}\sum_{j=1}^{\infty}\big(g_j^+\,\widetilde{g}_j^+ + g_j^-\,\widetilde{g}_j^-\big)=\tfrac{1}{2}\sum_{j=1}^{\infty}\left\langle\begin{pmatrix}g_j^+\\ g_j^-\end{pmatrix},\begin{pmatrix}\widetilde{g}_j^+\\ \widetilde{g}_j^-\end{pmatrix}\right\rangle_{\mathbb{R}^2}.\tag{2.17}$$

Let us now prove (2.15). First remark that, by construction, V is a subspace of $Y_{\mathbf{m}}^{s-1,\sigma}$ of dimension one. Therefore, we can apply the orthogonal supplementary theorem in the pre-Hilbertian vector space $\big(Y_{\mathbf{m}}^{s-1,\sigma},\langle\cdot,\cdot\rangle\big)$ to get

$$Y_{\mathbf{m}}^{s-1,\sigma}=V\overset{\perp}{\oplus}V^\perp.$$

This proves that $V^\perp$ is a subspace of $Y_{\mathbf{m}}^{s-1,\sigma}$ of codimension one. Since the range is also of codimension one in $Y_{\mathbf{m}}^{s-1,\sigma}$, it suffices to prove one inclusion is order to get the equality (2.15). For

$$g(x)=\sum_{j=1}^{\infty}M_{j\mathbf{m}}\big(a,b,c_{\mathbf{m}}^{\pm}(a,b)\big)\begin{pmatrix}h_j^+\\ h_j^-\end{pmatrix}\sin(2\pi j\mathbf{m}x)\in\mathcal{R}\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^{\pm}(a,b),0,0\big)\Big),$$

we have

$$\begin{aligned}\Big\langle g,g_{0,\mathbf{m},a,b}^{\pm}\Big\rangle&=\tfrac{1}{2}\left\langle M_{\mathbf{m}}\big(a,b,c_{\mathbf{m}}^{\pm}(a,b)\big)\begin{pmatrix}h_j^+\\ h_j^-\end{pmatrix},\begin{pmatrix}2\pi\mathbf{m}\big(a-c_{\mathbf{m}}^{\pm}(a,b)\big)-\frac{1}{2\pi\mathbf{m}(b-a)}\\ \frac{1}{2\pi\mathbf{m}(b-a)}\end{pmatrix}\right\rangle_{\mathbb{R}^2}\\&=\tfrac{1}{2}\left\langle\begin{pmatrix}h_j^+\\ h_j^-\end{pmatrix},M_{\mathbf{m}}^{\top}\big(a,b,c_{\mathbf{m}}^{\pm}(a,b)\big)\begin{pmatrix}2\pi\mathbf{m}\big(a-c_{\mathbf{m}}^{\pm}(a,b)\big)-\frac{1}{2\pi\mathbf{m}(b-a)}\\ \frac{1}{2\pi\mathbf{m}(b-a)}\end{pmatrix}\right\rangle_{\mathbb{R}^2}\\&=0.\end{aligned}$$

Here and in the sequel, the notation $M^\top$ refers to the transpose of the matrix M. The last equality is true because by construction

$$\begin{pmatrix}2\pi\mathbf{m}\big(a-c_{\mathbf{m}}^{\pm}(a,b)\big)-\frac{1}{2\pi\mathbf{m}(b-a)}\\ \frac{1}{2\pi\mathbf{m}(b-a)}\end{pmatrix}\in\ker\Big(M_{\mathbf{m}}^{\top}\big(a,b,c_{\mathbf{m}}^{\pm}(a,b)\big)\Big).$$

This proves

$$\mathcal{R}\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^{\pm}(a,b),0,0\big)\Big)\subset V^\perp.$$

Hence, by applying Lemma B.1, we conclude (2.15). Now, still for later puposes, we shall also show that

$$\mathcal{R}\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^{\pm}(a,b),0,0\big)\Big)=\ker(l),\tag{2.18}$$

with $l\in\big(Y_{\mathbf{m}}^{s-1,\sigma}\big)^*$ defined by

$$\forall g\in Y_{\mathbf{m}}^{s-1,\sigma},\quad l(g)\triangleq\Big\langle g,g_{0,\mathbf{m},a,b}^{\pm}\Big\rangle.$$

Actually, the identity (2.18) is just a reformulation of (2.15) and the only thing to prove is that $l\in\big(Y_{\mathbf{m}}^{s-1,\sigma}\big)^*$. For this aim, we denote $\|\cdot\|\triangleq\sqrt{\langle\cdot,\cdot\rangle}$ the norm associated with the scalar product $\langle\cdot,\cdot\rangle$. Then, for any $g=(g_+,g_-)\in Y_{\mathbf{m}}^{s-1,\sigma}$ in the form (2.16) (recall that $s\geqslant1$ and $\sigma>0$), we have

$$\begin{aligned}\|g\|^2&=\tfrac{1}{2}\sum_{j=1}^{\infty}\Big[\big(g_j^+\big)^2+\big(g_j^-\big)^2\Big]\\&\leqslant\tfrac{1}{2}\sum_{j=1}^{\infty}\Big[(2\pi j)^{2(s-1)}\big(g_j^+\big)^2e^{4\pi\sigma j}+(2\pi j)^{2(s-1)}\big(g_j^-\big)^2e^{4\pi\sigma j}\Big]\\&\leqslant\tfrac{1}{2}\big(\|g_+\|_{s-1,\sigma}^2+\|g_-\|_{s-1,\sigma}^2\big),\end{aligned}$$

from which we deduce

$$\begin{aligned}\|g\| &\leqslant \tfrac{1}{\sqrt{2}}\big(\|g_+\|_{s-1,\sigma}^2+\|g_-\|_{s-1,\sigma}^2\big)^{\frac{1}{2}}\\ &\leqslant \tfrac{1}{\sqrt{2}}\big(\|g_+\|_{s-1,\sigma}+\|g_-\|_{s-1,\sigma}\big)\\ &=\tfrac{1}{\sqrt{2}}\|g\|_{s-1,\sigma}.\end{aligned}$$

In the second inequality above, we have used the classical relation

$$\forall (a,b)\in(\mathbb{R}_+)^2,\quad \sqrt{a+b}\leqslant\sqrt{a}+\sqrt{b}.$$

Therefore, by Cauchy-Schwarz inequality, we get

$$\begin{aligned}|l(g)| &= \Big|\Big\langle g, g_{0,\mathbf{m},a,b}^{\pm}\Big\rangle\Big|\\ &\leqslant \Big\|g_{0,\mathbf{m},a,b}^{\pm}\Big\|\,\|g\|\\ &\leqslant \tfrac{1}{\sqrt{2}}\Big\|g_{0,\mathbf{m},a,b}^{\pm}\Big\|\,\|g\|_{s-1,\sigma}.\end{aligned}$$

This proves the claim.

► **Transversality condition :** From (2.5)-(2.6), we have

$$\partial_c d_{(\check{r}_+,\check{r}_-)}F(a,b,c,0,0)=\begin{pmatrix}-\partial_x & 0\\ 0 & -\partial_x\end{pmatrix}.$$

Hence,

$$\partial_c d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^{\pm}(a,b),0,0\big)[\check{r}_{0,\mathbf{m},a,b}^{\pm}](x)=2\pi\mathbf{m}\begin{pmatrix}2\pi\mathbf{m}\big(a-c_{\mathbf{m}}^{\pm}(a,b)\big)-\frac{1}{2\pi\mathbf{m}(b-a)}\\ -\frac{1}{2\pi\mathbf{m}(b-a)}\end{pmatrix}\sin(2\pi\mathbf{m}x).$$

Straightforward computations lead to

$$\Big\langle\partial_c d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^{\pm}(a,b),0,0\big)[\check{r}_{0,\mathbf{m},a,b}^{\pm}]\,,\,g_{0,\mathbf{m},a,b}^{\pm}\Big\rangle=2\pi\mathbf{m}\Big[a-c_{\mathbf{m}}^{\pm}(a,b)\Big]\Big[2\pi^2\mathbf{m}^2\big(a-c_{\mathbf{m}}^{\pm}(a,b)\big)-\tfrac{1}{b-a}\Big].\tag{2.19}$$

Since $c_{\mathbf{m}}^{+}(a,b)>b>a$, then

$$a-c_{\mathbf{m}}^{+}(a,b)<0\qquad\text{and}\qquad 2\pi^2\mathbf{m}^2\big(a-c_{\mathbf{m}}^{+}(a,b)\big)-\tfrac{1}{b-a}<0.$$

Thus,

$$\Big\langle\partial_c d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^{+}(a,b),0,0\big)[\check{r}_{0,\mathbf{m},a,b}^{+}]\,,\,g_{0,\mathbf{m},a,b}^{+}\Big\rangle>0.\tag{2.20}$$

In addition, one can easily check from (2.12) that

$$a>c_{\mathbf{m}}^{-}(a,b)>a-\frac{1}{2\pi^2\mathbf{m}^2(b-a)}.$$

Therefore,

$$\Big\langle\partial_c d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^{-}(a,b),0,0\big)[\check{r}_{0,\mathbf{m},a,b}^{-}]\,,\,g_{0,\mathbf{m},a,b}^{-}\Big\rangle<0.\tag{2.21}$$

In particular, in both cases, we have

$$\Big\langle\partial_c d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^{\pm}(a,b),0,0\big)[\check{r}_{0,\mathbf{m},a,b}^{\pm}]\,,\,g_{0,\mathbf{m},a,b}^{\pm}\Big\rangle\neq0,$$

which is, in view of (2.18), equivalent to

$$\partial_c d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^{\pm}(a,b),0,0\big)[\check{r}_{0,\mathbf{m},a,b}^{\pm}]\not\in\mathcal{R}\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c_{\mathbf{m}}^{\pm}(a,b),0,0\big)\Big).\tag{2.22}$$

Finally, (2.2), (2.4), (2.7), (2.14) and (2.22) allow to apply the Crandall-Rabinowitz Theorem A.1 which concludes the existence part of Theorem 1.1-(i). In the sequel, we denote

$$\mathscr{C}_{\text{local}}^{\pm,\mathbf{m}}(a,b):\mathbf{s}\in(-\delta,\delta)\mapsto\Big(c_{\mathbf{m}}^{\pm}(\mathbf{s},a,b)\,,\,\check{r}_{\mathbf{m}}^{\pm}(\mathbf{s},a,b)\Big)\in\mathbb{R}\times X_{\mathbf{m}}^{s,\sigma},\qquad\delta>0\tag{2.23}$$

the corresponding (real-analytic) local curve which satisfies

$$c_{\mathbf{m}}^{\pm}(0,a,b)=c_{\mathbf{m}}^{\pm}(a,b),\qquad\frac{d}{d\mathbf{s}}\Big(\check{r}_{\mathbf{m}}^{\pm}(\mathbf{s},a,b)\Big)\Big|_{\mathbf{s}=0}=\check{r}_{0,\mathbf{m},a,b}^{\pm}.\tag{2.24}$$

▶ **Pitchfork-type bifurcation :** According to Theorem A.1, we first need to prove that

$$d^2_{(\check{r}_+,\check{r}_-)}F\big(a,b,c^{\pm}_{\mathbf{m}}(a,b),0,0\big)[\check{r}^{\pm}_{0,\mathbf{m},a,b},\check{r}^{\pm}_{0,\mathbf{m},a,b}]\in\mathcal{R}\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c^{\pm}_{\mathbf{m}}(a,b),0,0\big)\Big).\tag{2.25}$$

One readily has from (2.1),

$$d^2_{(\check{r}_+,\check{r}_-)}F_{\pm}\big(a,b,c,0,0\big)[(h_+,h_-),(\widetilde{h}_+,\widetilde{h}_-)]=\partial_x\big(h_{\pm}\widetilde{h}_{\pm}\big),\qquad d^3_{(\check{r}_+,\check{r}_-)}F(a,b,c,0,0)=0.\tag{2.26}$$

Consequently, using the identity (2.3), we find

$$\begin{aligned}
&d^2_{(\check{r}_+,\check{r}_-)}F\big(a,b,c^{\pm}_{\mathbf{m}}(a,b),0,0\big)[\check{r}^{\pm}_{0,\mathbf{m},a,b},\check{r}^{\pm}_{0,\mathbf{m},a,b}]\\
&=\left(\frac{\Big(2\pi\mathbf{m}\big(a-c^{\pm}_{\mathbf{m}}(a,b)\big)-\frac{1}{2\pi\mathbf{m}(b-a)}\Big)^2}{\frac{1}{4\pi^2\mathbf{m}^2(b-a)^2}}\right)\partial_x\Big(\cos^2(2\pi\mathbf{m}x)\Big)\\
&=-4\pi\mathbf{m}\left(\frac{\Big(2\pi\mathbf{m}\big(a-c^{\pm}_{\mathbf{m}}(a,b)\big)-\frac{1}{2\pi\mathbf{m}(b-a)}\Big)^2}{\frac{1}{4\pi^2\mathbf{m}^2(b-a)^2}}\right)\sin(2\pi\mathbf{m}x)\cos(2\pi\mathbf{m}x)\\
&=-2\pi\mathbf{m}\left(\frac{\Big(2\pi\mathbf{m}\big(a-c^{\pm}_{\mathbf{m}}(a,b)\big)-\frac{1}{2\pi\mathbf{m}(b-a)}\Big)^2}{\frac{1}{4\pi^2\mathbf{m}^2(b-a)^2}}\right)\sin(4\pi\mathbf{m}x).
\end{aligned}\tag{2.27}$$

The orthogonality of the family $\big(x\mapsto\sin(2\pi j\mathbf{m}x)\big)_{j\in\mathbb{N}^*}$ with respect to the scalar product (2.17) implies

$$\Big\langle d^2_{(\check{r}_+,\check{r}_-)}F\big(a,b,c^{\pm}_{\mathbf{m}}(a,b),0,0\big)[\check{r}^{\pm}_{0,\mathbf{m},a,b},\check{r}^{\pm}_{0,\mathbf{m},a,b}]\,,\,g^{\pm}_{0,\mathbf{m},a,b}\Big\rangle=0,$$

which is equivalent to (2.25) according to (2.18). In addition, one can easily check from (2.8) and (2.27) that

$$\theta^{\pm}_{0,\mathbf{m},a,b}(x)\triangleq2\pi\mathbf{m}M^{-1}_{2\mathbf{m}}\big(a,b,c^{\pm}_{\mathbf{m}}(a,b)\big)\left(\frac{\Big(2\pi\mathbf{m}\big(a-c^{\pm}_{\mathbf{m}}(a,b)\big)-\frac{1}{2\pi\mathbf{m}(b-a)}\Big)^2}{\frac{1}{4\pi^2\mathbf{m}^2(b-a)^2}}\right)\cos(4\pi\mathbf{m}x)\tag{2.28}$$

satisfies

$$d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c^{\pm}_{\mathbf{m}}(a,b),0,0\big)[\theta^{\pm}_{0,\mathbf{m},a,b}]=d^2_{(\check{r}_+,\check{r}_-)}F\big(a,b,c^{\pm}_{\mathbf{m}}(a,b),0,0\big)[\check{r}^{\pm}_{0,\mathbf{m},a,b},\check{r}^{\pm}_{0,\mathbf{m},a,b}].$$

Then, according to Theorem A.1, (2.25) and (2.26), we have

$$\frac{d}{d\mathbf{s}}\Big(c^{\pm}_{\mathbf{m}}(\mathbf{s},a,b)\Big)\Big|_{\mathbf{s}=0}=0\tag{2.29}$$

and

$$\frac{d^2}{d\mathbf{s}^2}\Big(c^{\pm}_{\mathbf{m}}(\mathbf{s},a,b)\Big)\Big|_{\mathbf{s}=0}=\frac{\Big\langle d^2_{(\check{r}_+,\check{r}_-)}F\big(a,b,c^{\pm}_{\mathbf{m}}(a,b),0,0\big)[\check{r}^{\pm}_{0,\mathbf{m},a,b},\theta^{\pm}_{0,\mathbf{m},a,b}]\,,\,g^{\pm}_{0,\mathbf{m},a,b}\Big\rangle}{\Big\langle\partial_c d_{(\check{r}_+,\check{r}_-)}F\big(a,b,c^{\pm}_{\mathbf{m}}(a,b),0,0\big)[\check{r}^{\pm}_{0,\mathbf{m},a,b}]\,,\,g^{\pm}_{0,\mathbf{m},a,b}\Big\rangle}.\tag{2.30}$$

Now, to conclude the pitchfork bifurcation, it remains to prove that

$$\Big\langle d^2_{(\check{r}_+,\check{r}_-)}F\big(a,b,c^{\pm}_{\mathbf{m}}(a,b),0,0\big)[\check{r}^{\pm}_{0,\mathbf{m},a,b},\theta^{\pm}_{0,\mathbf{m},a,b}]\,,\,g^{\pm}_{0,\mathbf{m},a,b}\Big\rangle\neq0$$

and more precisely, to study the sign of the previous quantity in order to obtain the (local) direction of the branch. Looking at (2.9), we denote for any $j\in\mathbb{N}^*$,

$$M_j\big(a,b,c\big)\triangleq\begin{pmatrix}\alpha_j\big(a,b,c\big)&-\gamma_j(a,b)\\\gamma_j(a,b)&\beta_j\big(a,b,c\big)\end{pmatrix}.\tag{2.31}$$

With this notation, we have

$$\check{r}^{\pm}_{0,\mathbf{m},a,b}(x)=\begin{pmatrix}\beta_{\mathbf{m}}\big(a,b,c^{\pm}_{\mathbf{m}}(a,b)\big)\\-\gamma_{\mathbf{m}}(a,b)\end{pmatrix}\cos(2\pi\mathbf{m}x),\qquad g^{\pm}_{0,\mathbf{m},a,b}(x)=\begin{pmatrix}\beta_{\mathbf{m}}\big(a,b,c^{\pm}_{\mathbf{m}}(a,b)\big)\\\gamma_{\mathbf{m}}(a,b)\end{pmatrix}\sin(2\pi\mathbf{m}x)$$

and, using (2.13) and (2.28),

$$\begin{aligned}
\theta^{\pm}_{0,\mathbf{m},a,b}(x)&=\frac{2\pi\mathbf{m}}{3}\begin{pmatrix}\beta_{2\mathbf{m}}\big(a,b,c^{\pm}_{\mathbf{m}}(a,b)\big)&\gamma_{2\mathbf{m}}(a,b)\\-\gamma_{2\mathbf{m}}(a,b)&\alpha_{2\mathbf{m}}\big(a,b,c^{\pm}_{\mathbf{m}}(a,b)\big)\end{pmatrix}\begin{pmatrix}\beta^2_{\mathbf{m}}\big(a,b,c^{\pm}_{\mathbf{m}}(a,b)\big)\\\gamma^2_{\mathbf{m}}(a,b)\end{pmatrix}\cos(4\pi\mathbf{m}x)\\
&=\frac{2\pi\mathbf{m}}{3}\begin{pmatrix}\beta_{2\mathbf{m}}\big(a,b,c^{\pm}_{\mathbf{m}}(a,b)\big)\beta^2_{\mathbf{m}}\big(a,b,c^{\pm}_{\mathbf{m}}(a,b)\big)+\gamma_{2\mathbf{m}}(a,b)\gamma^2_{\mathbf{m}}(a,b)\\-\gamma_{2\mathbf{m}}(a,b)\beta^2_{\mathbf{m}}\big(a,b,c^{\pm}_{\mathbf{m}}(a,b)\big)+\alpha_{2\mathbf{m}}\big(a,b,c^{\pm}_{\mathbf{m}}(a,b)\big)\gamma^2_{\mathbf{m}}(a,b)\end{pmatrix}\cos(4\pi\mathbf{m}x).
\end{aligned}$$

Therefore, by (2.26), we infer

$$\begin{aligned}
&d^2_{(\check r_+,\check r_-)}F\big(a,b,c^{\pm}_{\mathbf m}(a,b),0,0\big)[\check r^{\pm}_{0,\mathbf m,a,b},\theta^{\pm}_{0,\mathbf m,a,b}]\\
&=\tfrac{2\pi\mathbf m\partial_x\left(\cos(4\pi\mathbf m x)\cos(2\pi\mathbf m x)\right)}{3}\begin{pmatrix}\beta_{2\mathbf m}\big(a,b,c^{\pm}_{\mathbf m}(a,b)\big)\beta^3_{\mathbf m}\big(a,b,c^{\pm}_{\mathbf m}(a,b)\big)+\gamma_{2\mathbf m}(a,b)\gamma^2_{\mathbf m}(a,b)\beta_{\mathbf m}\big(a,b,c^{\pm}_{\mathbf m}(a,b)\big)\\ -\alpha_{2\mathbf m}\big(a,b,c^{\pm}_{\mathbf m}(a,b)\big)\gamma^3_{\mathbf m}(a,b)+\gamma_{2\mathbf m}(a,b)\gamma_{\mathbf m}(a,b)\beta^2_{\mathbf m}\big(a,b,c^{\pm}_{\mathbf m}(a,b)\big)\end{pmatrix}.
\end{aligned}$$

Now, from the classical relation

$$\forall(u,v)\in\mathbb R^2,\quad \cos(u)\cos(v)=\tfrac12\big(\cos(u+v)+\cos(u-v)\big),$$

we obtain

$$\partial_x\Big(\cos(4\pi\mathbf m x)\cos(2\pi\mathbf m x)\Big)=-3\pi\mathbf m\sin(6\pi\mathbf m x)-\pi\mathbf m\sin(2\pi\mathbf m x).$$

Together with the orthogonality of the family $\big(x\mapsto\sin(2\pi j\mathbf m x)\big)_{j\in\mathbb N^*}$ with respect to the scalar product (2.17), we get

$$\Big\langle d^2_{(\check r_+,\check r_-)}F\big(a,b,c^{\pm}_{\mathbf m}(a,b),0,0\big)[\check r^{\pm}_{0,\mathbf m,a,b},\theta^{\pm}_{0,\mathbf m,a,b}]\,,\,g^{\pm}_{0,\mathbf m,a,b}\Big\rangle=\frac{\pi^2\mathbf m^2}{3}\mathtt f^{\pm}(\mathbf m,a,b),\tag{2.32}$$

where $\mathtt f^{\pm}$ is a well-defined continuous fonction on $[1,\infty)\times\mathbb S$ given by

$$\mathtt f^{\pm}(z,a,b)\triangleq\alpha_{2z}\big(a,b,c^{\pm}_z(a,b)\big)\gamma^4_z(a,b)-\beta_{2z}\big(a,b,c^{\pm}_z(a,b)\big)\beta^4_z\big(a,b,c^{\pm}_z(a,b)\big)-2\gamma_{2z}(a,b)\gamma^2_z(a,b)\beta^2_z\big(a,b,c^{\pm}_z(a,b)\big).$$

After tedious computations using (2.12), we find

$$\mathtt f^{\pm}(z,a,b)=\frac{1}{4\pi^3z^3(b-a)^3}\Big(\pm A(z,a,b)\sqrt{\pi^2z^2(b-a)^2+1}+B(z,a,b)\Big),\tag{2.33}$$

where

$$\begin{aligned}
A(z,a,b)&\triangleq 128\pi^7z^7(b-a)^7+232\pi^5z^5(b-a)^5+132\pi^3z^3(b-a)^3+24\pi z(b-a),\\
B(z,a,b)&\triangleq 128\pi^8z^8(b-a)^8+296\pi^6z^6(b-a)^6+232\pi^4z^4(b-a)^4+69\pi^2z^2(b-a)^2+6.
\end{aligned}$$

Clearly, with this expression, we can conclude that

$$\forall(z,a,b)\in[1,\infty)\times\mathbb S,\quad \mathtt f^{+}(z,a,b)>0.$$

Combined with (2.32), we deduce

$$\Big\langle d^2_{(\check r_+,\check r_-)}F\big(a,b,c^{+}_{\mathbf m}(a,b),0,0\big)[\check r^{+}_{0,\mathbf m,a,b},\theta^{+}_{0,\mathbf m,a,b}]\,,\,g^{+}_{0,\mathbf m,a,b}\Big\rangle>0.\tag{2.34}$$

Assume, for the sake of contradiction, that there exists $(z,a,b)\in[1,\infty)\times\mathbb S$ such that $\mathtt f^{-}(z,a,b)=0$. Then this equation is equivalent to

$$\sqrt{\pi^2z^2(b-a)^2+1}=\frac{B(z,a,b)}{A(z,a,b)}.$$

Taking the square of the previous expression, then, after simplifications, we end up with

$$0=256\pi^8z^8(b-a)^8+672\pi^6z^6(b-a)^6+633\pi^4z^4(b-a)^4+252\pi^2z^2(b-a)^2+36,$$

which is impossible since the right hand-side is positive. We deduce that

$$\forall(z,a,b)\in[1,\infty)\times\mathbb S,\quad \mathtt f^{-}(z,a,b)\neq0.$$

Now, we have the following asymptotic

$$\mathtt f^{-}(z,a,b)\underset{z\to\infty}{\sim}16\pi^3z^3(b-a)^3.$$

Hence, by a continuity argument, we infer

$$\forall(z,a,b)\in[1,\infty)\times\mathbb S,\quad \mathtt f^{-}(z,a,b)>0.$$

Combined with (2.32), we deduce

$$\Big\langle d^2_{(\check r_+,\check r_-)}F\big(a,b,c^{-}_{\mathbf m}(a,b),0,0\big)[\check r^{-}_{0,\mathbf m,a,b},\theta^{-}_{0,\mathbf m,a,b}]\,,\,g^{-}_{0,\mathbf m,a,b}\Big\rangle>0.\tag{2.35}$$

Finally, putting together (2.30), (2.20), (2.21), (2.34) and (2.35), we obtain

$$\frac{d^2}{d\mathtt s^2}\Big(c^{-}_{\mathbf m}(\mathtt s,a,b)\Big)\Big|_{\mathtt s=0}<0\qquad\text{and}\qquad\frac{d^2}{d\mathtt s^2}\Big(c^{+}_{\mathbf m}(\mathtt s,a,b)\Big)\Big|_{\mathtt s=0}>0,$$

which corresponds to a "hyperbolic" bifurcation diagram. This achieves the proof of Theorem 1.1-(i).

2.2 Bifurcation from b

In this subsection, we fix $a, c \in \mathbb{R}$ with $a \neq c$ and study the bifurcation with respect to the parameter $b > a$. Observe that (2.10) implies

$$\Delta_j(a,b,c) = 0 \qquad \Leftrightarrow \qquad b = b_j(a,c) \triangleq c + \frac{1}{4\pi^2 j^2 (a-c)}\cdot \tag{2.36}$$

Here and in the sequel, we shall use the following notation: for any $p \in \mathbb{R}^*$, we denote

$$N_1(p) \triangleq 1 + N_2(p), \qquad N_2(p) \triangleq \left\lfloor \tfrac{1}{2\pi|p|} \right\rfloor.$$

The constraint $b > a$ requires the following restriction

$$\begin{cases} 4\pi^2 j^2 (a-c)^2 > 1, & \text{if } a < c, \\ 4\pi^2 j^2 (a-c)^2 < 1, & \text{if } a > c. \end{cases} \tag{2.37}$$

`Case 1` : If $a < c$, then there is a countable family of potential bifurcation points, namely

$$b_{\mathbf{m}}(a,c), \qquad \mathbf{m} \geqslant N_1(c-a).$$

`Case 2` : If $a > c$, then there are two options.

- If $a \geqslant c + \frac{1}{2\pi}$, no bifurcation point exists.
- If $a < c + \frac{1}{2\pi}$, then there is a finite number of potential bifurcation points, namely

$$b_{\mathbf{m}}(a,c), \qquad \mathbf{m} \in [\![1, N_2(a-c)]\!].$$

In what follows, the condition $a > c$ always refers to $c < a < c + \frac{1}{2\pi}\cdot$

► **One dimensional kernel condition :** For $a < c$ (resp. $a > c$) the sequence $\big(b_j(a,c)\big)_{j\in\mathbb{N}^*}$ (well-defined) is increasing (resp. decreasing) and tends to c as $j \to \infty$. In addition, one has, for any fixed $\mathbf{m} \in \mathbb{N}^*$ with $\mathbf{m} \geqslant N_1(c-a)$ (resp. $\mathbf{m} \in [\![1, N_2(a-c)]\!]$),

$$\Delta_{\mathbf{m}}\big(a, b_{\mathbf{m}}(a,c), c\big) = 0$$

and, similarly to (2.13),

$$\forall j \in \mathbb{N} \setminus \{0,1\}, \quad \Delta_{j\mathbf{m}}\big(a, b_{\mathbf{m}}(a,c), c\big) = j^2 - 1 > 0. \tag{2.38}$$

Thus, in both cases, the kernel of $d_{(\check{r}_+,\check{r}_-)}F\big(a, b_{\mathbf{m}}(a,c), c, 0, 0\big)$ is one dimensional, more precisely

$$\begin{aligned} &\ker\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a, b_{\mathbf{m}}(a,c), c, 0, 0\big)\Big) = \mathtt{span}(\check{r}_{0,\mathbf{m},a,c}), \\ &\check{r}_{0,\mathbf{m},a,c}(x) \triangleq \begin{pmatrix} 2\pi\mathbf{m}\big(a-c\big) - \frac{1}{2\pi\mathbf{m}\left(b_{\mathbf{m}}(a,c)-a\right)} \\ -\frac{1}{2\pi\mathbf{m}\left(b_{\mathbf{m}}(a,c)-a\right)} \end{pmatrix} \cos(2\pi\mathbf{m}x). \end{aligned} \tag{2.39}$$

► **Range condition :** In both cases, $a \neq c$ and the definition of $b_{\mathbf{m}}(a,c)$ in (2.36) implies that $b_{\mathbf{m}}(a,c) \neq c$. So the Fredholmness property (2.7) gives that the range $\mathcal{R}\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a, b_{\mathbf{m}}(a,c), c, 0, 0\big)\Big)$ is closed and of codimension one in $Y^{s-1,\sigma}_{\mathbf{m}}$. More precisely, arguing as in the previous subsection, we find

$$\begin{aligned} &\mathcal{R}\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a, b_{\mathbf{m}}(a,c), c, 0, 0\big)\Big) = \Big(\mathtt{span}(g_{0,\mathbf{m},a,c})\Big)^{\perp} = \ker\Big(g \mapsto \langle g, g_{0,\mathbf{m},a,c}\rangle\Big), \\ &g_{0,\mathbf{m},a,c}(x) \triangleq \begin{pmatrix} 2\pi\mathbf{m}\big(a-c\big) - \frac{1}{2\pi\mathbf{m}\left(b_{\mathbf{m}}(a,c)-a\right)} \\ \frac{1}{2\pi\mathbf{m}\left(b_{\mathbf{m}}(a,c)-a\right)} \end{pmatrix} \sin(2\pi\mathbf{m}x), \end{aligned} \tag{2.40}$$

where the orthogonal is understood in the sense of the scalar product defined in (2.17).

► **Transversality condition :** From (2.5)-(2.6), we have

$$\partial_b d_{(\check{r}_+,\check{r}_-)}F(a,b,c,0,0) = \begin{pmatrix} \partial_x & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{(b-a)^2}\begin{pmatrix} \partial_x^{-1} & -\partial_x^{-1} \\ \partial_x^{-1} & -\partial_x^{-1} \end{pmatrix}.$$

Hence,

$$\partial_b d_{(\check r_+,\check r_-)}F\big(a,b_{\mathbf m}(a,c),c,0,0\big)[\check r_{0,\mathbf m,a,c}]=h^{(2)}_{0,\mathbf m,a,c}-h^{(1)}_{0,\mathbf m,a,c},$$

where

$$h^{(1)}_{0,\mathbf m,a,c}(x)\triangleq 2\pi\mathbf m\begin{pmatrix}2\pi\mathbf m(a-c)-\frac{1}{2\pi\mathbf m\big(b_{\mathbf m}(a,c)-a\big)}\\ 0\end{pmatrix}\sin(2\pi\mathbf m x),$$

$$h^{(2)}_{0,\mathbf m,a,c}(x)\triangleq\frac{a-c}{\big(b_{\mathbf m}(a,c)-a\big)^2}\begin{pmatrix}1\\1\end{pmatrix}\sin(2\pi\mathbf m x).$$

On one hand

$$\Big\langle h^{(1)}_{0,\mathbf m,a,c}\,,\,g_{0,\mathbf m,a,c}\Big\rangle=\pi\mathbf m\left(2\pi\mathbf m(a-c)-\frac{1}{2\pi\mathbf m\big(b_{\mathbf m}(a,c)-a\big)}\right)^2.$$

On the other hand

$$\Big\langle h^{(2)}_{0,\mathbf m,a,c}\,,\,g_{0,\mathbf m,a,c}\Big\rangle=\frac{\pi\mathbf m(a-c)^2}{\big(b_{\mathbf m}(a,c)-a\big)^2}.$$

Hence, using (2.36), we obtain

$$\begin{aligned}&\Big\langle\partial_b d_{(\check r_+,\check r_-)}F\big(a,b_{\mathbf m}(a,c),c,0,0\big)[\check r_{0,\mathbf m,a,c}]\,,\,g_{0,\mathbf m,a,c}\Big\rangle=\Big\langle h^{(2)}_{0,\mathbf m,a,c}-h^{(1)}_{0,\mathbf m,a,c}\,,\,g_{0,\mathbf m,a,c}\Big\rangle\\&=\pi\mathbf m\left(\frac{(a-c)^2}{\big(b_{\mathbf m}(a,c)-a\big)^2}-4\pi^2\mathbf m^2(a-c)^2+2\frac{a-c}{b_{\mathbf m}(a,c)-a}-\frac{1}{4\pi^2\mathbf m^2\big(b_{\mathbf m}(a,c)-a\big)^2}\right)\\&=\frac{\pi\mathbf m(a-c)^2}{\big(b_{\mathbf m}(a,c)-a\big)^2}\Big(1-4\pi^2\mathbf m^2(a-c)^2\Big).\end{aligned}$$

Therefore, the condition (2.37) gives

$$\begin{cases}\Big\langle\partial_b d_{(\check r_+,\check r_-)}F\big(a,b_{\mathbf m}(a,c),c,0,0\big)[\check r_{0,\mathbf m,a,c}]\,,\,g_{0,\mathbf m,a,c}\Big\rangle<0, & \text{if } a<c,\\ \Big\langle\partial_b d_{(\check r_+,\check r_-)}F\big(a,b_{\mathbf m}(a,c),c,0,0\big)[\check r_{0,\mathbf m,a,c}]\,,\,g_{0,\mathbf m,a,c}\Big\rangle>0, & \text{if } a>c.\end{cases}\tag{2.41}$$

In both cases

$$\Big\langle\partial_b d_{(\check r_+,\check r_-)}F\big(a,b_{\mathbf m}(a,c),c,0,0\big)[\check r_{0,\mathbf m,a,c}]\,,\,g_{0,\mathbf m,a,c}\Big\rangle\neq 0,$$

which means

$$\partial_b d_{(\check r_+,\check r_-)}F\big(a,b_{\mathbf m}(a,c),c,0,0\big)[\check r_{0,\mathbf m,a,c}]\not\in\mathcal R\Big(d_{(\check r_+,\check r_-)}F\big(a,b_{\mathbf m}(a,c),c,0,0\big)\Big).\tag{2.42}$$

Finally, (2.2), (2.4), (2.7), (2.39) and (2.42) allow to apply the Crandall-Rabinowitz Theorem A.1 which concludes the existence part of Theorem 1.1-(ii). In the sequel, we denote

$$\mathscr C^{\mathbf m}_{\text{local}}(a,c):\mathtt s\in(-\delta,\delta)\mapsto\Big(b_{\mathbf m}(\mathtt s,a,c)\,,\,\check r_{\mathbf m}(\mathtt s,a,c)\Big)\in\mathbb R\times X^{s,\sigma}_{\mathbf m},\qquad\delta>0\tag{2.43}$$

the corresponding (real-analytic) local curve which satisfies

$$b_{\mathbf m}(0,a,c)=b_{\mathbf m}(a,c),\qquad\frac{d}{d\mathtt s}\Big(\check r_{\mathbf m}(\mathtt s,a,c)\Big)\Big|_{\mathtt s=0}=\check r_{0,\mathbf m,a,c}.\tag{2.44}$$

► **Pitchfork-type bifurcation :** As in the previous subsection, we can write

$$d^2_{(\check r_+,\check r_-)}F\big(a,b_{\mathbf m}(a,c),c,0,0\big)[\check r_{0,\mathbf m,a,c},\check r_{0,\mathbf m,a,c}]=d_{(\check r_+,\check r_-)}F\big(a,b_{\mathbf m}(a,c),c,0,0\big)[\theta_{0,\mathbf m,a,c}],$$

with

$$\theta_{0,\mathbf m,a,c}(x)\triangleq 2\pi\mathbf m M^{-1}_{2\mathbf m}\big(a,b_{\mathbf m}(a,c),c\big)\begin{pmatrix}\left(2\pi\mathbf m(a-c)-\frac{1}{2\pi\mathbf m\big(b_{\mathbf m}(a,c)-a\big)}\right)^2\\ \frac{1}{4\pi^2\mathbf m^2\big(b_{\mathbf m}(a,c)-a\big)^2}\end{pmatrix}\cos(4\pi\mathbf m x).$$

Therefore, Theorem A.1 applies and gives

$$\frac{d}{d\mathtt s}\Big(b_{\mathbf m}(\mathtt s,a,c)\Big)\Big|_{\mathtt s=0}=0$$

and

$$\frac{d^2}{d\mathtt{s}^2}\Big(b_{\mathbf{m}}(\mathtt{s},a,c)\Big)\Big|_{\mathtt{s}=0}=\frac{\Big\langle d^2_{(\check{r}_+,\check{r}_-)}F\big(a,b_{\mathbf{m}}(a,c),c,0,0\big)[\check{r}_{0,\mathbf{m},a,c},\theta_{0,\mathbf{m},a,c}]\,,\,g_{0,\mathbf{m},a,c}\Big\rangle}{\big\langle\partial_b d_{(\check{r}_+,\check{r}_-)}F\big(a,b_{\mathbf{m}}(a,c),c,0,0\big)[\check{r}_{0,\mathbf{m},a,c}]\,,\,g_{0,\mathbf{m},a,c}\big\rangle}\cdot \tag{2.45}$$

In addition,

$$\Big\langle d^2_{(\check{r}_+,\check{r}_-)}F\big(a,b_{\mathbf{m}}(a,c),c,0,0\big)[\check{r}_{0,\mathbf{m},a,c},\theta_{0,\mathbf{m},a,c}]\,,\,g_{0,\mathbf{m},a,c}\Big\rangle=\frac{\pi^2\mathbf{m}^2}{3}\mathtt{h}(\mathbf{m},a,c),$$

where

$$\begin{aligned}\mathtt{h}(\mathbf{m},a,c)\triangleq{}&\alpha_{2\mathbf{m}}\big(a,b_{\mathbf{m}}(a,c),c\big)\gamma^4_{\mathbf{m}}\big(a,b_{\mathbf{m}}(a,c)\big)-\beta_{2\mathbf{m}}\big(a,b_{\mathbf{m}}(a,c),c\big)\beta^4_{\mathbf{m}}\big(a,b_{\mathbf{m}}(a,c),c\big)\\&-2\gamma_{2\mathbf{m}}\big(a,b_{\mathbf{m}}(a,c)\big)\gamma^2_{\mathbf{m}}\big(a,b_{\mathbf{m}}(a,c)\big)\beta^2_{\mathbf{m}}\big(a,b_{\mathbf{m}}(a,c),c\big).\end{aligned}$$

After tedious calculations using (2.36), we can write

$$\mathtt{h}(\mathbf{m},a,c)=\frac{4\pi^3\mathbf{m}^3(a-c)^3}{\big(1-4\pi^2\mathbf{m}^2(a-c)^2\big)^5}\Big(\mathtt{h}_1\big(4\pi^2\mathbf{m}^2(a-c)^2\big)+4\Big),$$

with

$$\forall x\geqslant 0,\quad \mathtt{h}_1(x)\triangleq 4x^6-3x^5-2x^3-3x. \tag{2.46}$$

Observe that

$$\forall x\geqslant 0,\quad \mathtt{h}_1'(x)=24x^5-15x^4-6x^2-3=3(x-1)(8x^4+3x^3+3x^2+x+1).$$

We deduce that the fonction $\mathtt{h}_1$ is decreasing on $[0,1]$, increasing on $[1,\infty)$ and admits a global minimum on $[0,\infty)$ at $x=1$ with value $\mathtt{h}_1(1)=-4$. Therefore, together with the constraint (2.37), we deduce that

$$\mathtt{h}(\mathbf{m},a,c)>0.$$

This implies in turn

$$\Big\langle d^2_{(\check{r}_+,\check{r}_-)}F\big(a,b_{\mathbf{m}}(a,c),c,0,0\big)[\check{r}_{0,\mathbf{m},a,c},\theta_{0,\mathbf{m},a,c}]\,,\,g_{0,\mathbf{m},a,c}\Big\rangle>0. \tag{2.47}$$

Combining (2.45), (2.41) and (2.47), we obtain

$$\begin{cases}\frac{d^2}{d\mathtt{s}^2}\Big(b_{\mathbf{m}}(\mathtt{s},a,c)\Big)\Big|_{\mathtt{s}=0}<0, & \text{if } a<c,\\ \frac{d^2}{d\mathtt{s}^2}\Big(b_{\mathbf{m}}(\mathtt{s},a,c)\Big)\Big|_{\mathtt{s}=0}>0, & \text{if } a>c.\end{cases}$$

This concludes the proof of Theorem 1.1-(ii).

2.3 Bifurcation from a

In this subsection, we fix $b,c\in\mathbb{R}$ with $b\neq c$ and study the bifurcation with respect to the parameter $a<b$. This subsection is very similar to the previous one, but cannot be reduced by a symmetry argument because of the transversality and pitchfork analysis. Observe that (2.10) implies

$$\Delta_j(a,b,c)=0\qquad\Leftrightarrow\qquad a=a_j(b,c)\triangleq c+\frac{1}{4\pi^2j^2(b-c)}\cdot \tag{2.48}$$

The constraint $a<b$ requires the following restriction

$$\begin{cases}4\pi^2j^2(b-c)^2>1, & \text{if } b>c,\\ 4\pi^2j^2(b-c)^2<1, & \text{if } b<c.\end{cases} \tag{2.49}$$

`Case 1 :` If $b>c$, then there is a countable family of potential bifurcation points, namely

$$a_{\mathbf{m}}(b,c),\qquad \mathbf{m}\geqslant N_1(b-c).$$

`Case 2 :` If $b<c$, then there are two options.

- If $b\leqslant c-\frac{1}{2\pi}$, no bifurcation point exists.
- If $b>c-\frac{1}{2\pi}$ and there is a finite number of potential bifurcation points, namely

$$a_{\mathbf{m}}(b,c),\qquad \mathbf{m}\in[\![1,N_2(c-b)]\!].$$

In what follows, the condition $b<c$ always refers to $c-\frac{1}{2\pi}<b<c$.

▶ **One dimensional kernel condition :** For $b>c$ (resp. $b<c$), the sequence $\big(a_j(b,c)\big)_{j\in\mathbb{N}^*}$ (well-defined) is decreasing (resp. increasing) and tends to c as $j\to\infty$. In addition, one has, for any fixed $\mathbf{m}\in\mathbb{N}^*$ with $\mathbf{m}\geqslant N_1(b-c)$ (resp. $\mathbf{m}\in[\![1,N_2(c-b)]\!]$),

$$\Delta_{\mathbf{m}}\big(a_{\mathbf{m}}(b,c),b,c\big)=0$$

and, similarly to (2.13),

$$\forall j\in\mathbb{N}\setminus\{0,1\},\quad \Delta_{j\mathbf{m}}\big(a_{\mathbf{m}}(b,c),b,c\big)=j^2-1. \tag{2.50}$$

Thus, in both cases, the kernel of $d_{(\check{r}_+,\check{r}_-)}F\big(a_{\mathbf{m}}(b,c),b,c,0,0\big)$ is one dimensional, more precisely

$$\begin{aligned}&\ker\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a_{\mathbf{m}}(b,c),b,c,0,0\big)\Big)=\mathtt{span}(\check{r}_{0,\mathbf{m},b,c}),\\ &\check{r}_{0,\mathbf{m},b,c}(x)\triangleq\begin{pmatrix}2\pi\mathbf{m}\big(a_{\mathbf{m}}(b,c)-c\big)-\frac{1}{2\pi\mathbf{m}\big(b-a_{\mathbf{m}}(b,c)\big)}\\ -\frac{1}{2\pi\mathbf{m}\big(b-a_{\mathbf{m}}(b,c)\big)}\end{pmatrix}\cos(2\pi\mathbf{m}x).\end{aligned} \tag{2.51}$$

▶ **Range condition :** In both cases, $b\neq c$ and the definition of $a_{\mathbf{m}}(b,c)$ in (2.48) implies that $a_{\mathbf{m}}(b,c)\neq c$. So the Fredholmness property (2.7) gives that the range $\mathcal{R}\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a_{\mathbf{m}}(b,c),b,c,0,0\big)\Big)$ is closed and of codimension one in $Y_{\mathbf{m}}^{s-1,\sigma}$. More precisely, arguing as in the subsection 2.1, we find

$$\begin{aligned}&\mathcal{R}\Big(d_{(\check{r}_+,\check{r}_-)}F\big(a_{\mathbf{m}}(b,c),b,c,0,0\big)\Big)=\Big(\mathtt{span}(g_{0,\mathbf{m},b,c})\Big)^{\perp}=\ker\Big(g\mapsto\langle g,g_{0,\mathbf{m},b,c}\rangle\Big),\\ &g_{0,\mathbf{m},b,c}(x)\triangleq\begin{pmatrix}2\pi\mathbf{m}\big(a_{\mathbf{m}}(b,c)-c\big)-\frac{1}{2\pi\mathbf{m}\big(b-a_{\mathbf{m}}(b,c)\big)}\\ \frac{1}{2\pi\mathbf{m}\big(b-a_{\mathbf{m}}(b,c)\big)}\end{pmatrix}\sin(2\pi\mathbf{m}x),\end{aligned} \tag{2.52}$$

where the orthogonal is understood in the sense of the scalar product defined in (2.17).
▶ **Transversality condition :** From (2.5)-(2.6), we have

$$\partial_a d_{(\check{r}_+,\check{r}_-)}F(a,b,c,0,0)=\begin{pmatrix}0&0\\0&\partial_x\end{pmatrix}+\frac{1}{(b-a)^2}\begin{pmatrix}-\partial_x^{-1}&\partial_x^{-1}\\-\partial_x^{-1}&\partial_x^{-1}\end{pmatrix}.$$

Hence,

$$\partial_a d_{(\check{r}_+,\check{r}_-)}F\big(a_{\mathbf{m}}(b,c),b,c,0,0\big)[\check{r}_{0,\mathbf{m},b,c}]=h^{(1)}_{0,\mathbf{m},b,c}-h^{(2)}_{0,\mathbf{m},b,c},$$

where

$$h^{(1)}_{0,\mathbf{m},b,c}(x)\triangleq\begin{pmatrix}0\\ \frac{1}{b-a_{\mathbf{m}}(b,c)}\end{pmatrix}\sin(2\pi\mathbf{m}x),\qquad h^{(2)}_{0,\mathbf{m},b,c}(x)\triangleq\frac{a_{\mathbf{m}}(b,c)-c}{\big(b-a_{\mathbf{m}}(b,c)\big)^2}\begin{pmatrix}1\\1\end{pmatrix}\sin(2\pi\mathbf{m}x).$$

We have

$$\Big\langle h^{(1)}_{0,\mathbf{m},b,c}\,,\,g_{0,\mathbf{m},b,c}\Big\rangle=\frac{1}{4\pi\mathbf{m}\big(b-a_{\mathbf{m}}(b,c)\big)^2},\qquad \Big\langle h^{(2)}_{0,\mathbf{m},b,c}\,,\,g_{0,\mathbf{m},b,c}\Big\rangle=\frac{\pi\mathbf{m}\big(a_{\mathbf{m}}(b,c)-c\big)^2}{\big(b-a_{\mathbf{m}}(b,c)\big)^2}.$$

Hence, using (2.48), we obtain

$$\begin{aligned}&\big\langle\partial_a d_{(\check{r}_+,\check{r}_-)}F\big(a_{\mathbf{m}}(b,c),b,c,0,0\big)[\check{r}_{0,\mathbf{m},b,c}]\,,\,g_{0,\mathbf{m},b,c}\big\rangle=\Big\langle h^{(1)}_{0,\mathbf{m},b,c}-h^{(2)}_{0,\mathbf{m},b,c}\,,\,g_{0,\mathbf{m},b,c}\Big\rangle\\ &=\frac{1}{4\pi\mathbf{m}\big(b-a_{\mathbf{m}}(b,c)\big)^2}\Big(1-4\pi^2\mathbf{m}^2\big(a_{\mathbf{m}}(b,c)-c\big)^2\Big)\\ &=\frac{1}{16\pi^3\mathbf{m}^3(b-c)^2\big(b-a_{\mathbf{m}}(b,c)\big)^2}\Big(4\pi^2\mathbf{m}^2(b-c)^2-1\Big).\end{aligned}$$

Therefore, the condition (2.49) gives

$$\begin{cases}\big\langle\partial_a d_{(\check{r}_+,\check{r}_-)}F\big(a_{\mathbf{m}}(b,c),b,c,0,0\big)[\check{r}_{0,\mathbf{m},b,c}]\,,\,g_{0,\mathbf{m},b,c}\big\rangle>0, & \text{if } b>c,\\ \big\langle\partial_a d_{(\check{r}_+,\check{r}_-)}F\big(a_{\mathbf{m}}(b,c),b,c,0,0\big)[\check{r}_{0,\mathbf{m},b,c}]\,,\,g_{0,\mathbf{m},b,c}\big\rangle<0, & \text{if } b<c.\end{cases} \tag{2.53}$$

In both cases

$$\big\langle\partial_a d_{(\check{r}_+,\check{r}_-)}F\big(a_{\mathbf{m}}(b,c),b,c,0,0\big)[\check{r}_{0,\mathbf{m},b,c}]\,,\,g_{0,\mathbf{m},b,c}\big\rangle\neq0,$$

which means
$$\partial_a d_{(\check r_+,\check r_-)}F\big(a_{\mathbf m}(b,c),b,c,0,0\big)[\check r_{0,\mathbf m,b,c}]\notin\mathcal R\Big(d_{(\check r_+,\check r_-)}F\big(a_{\mathbf m}(b,c),b,c,0,0\big)\Big).\tag{2.54}$$
Finally, (2.2), (2.4), (2.7), (2.51) and (2.54) allow to apply the Crandall-Rabinowitz Theorem A.1 which concludes the existence part of Theorem 1.1-(iii). In the sequel, we denote
$$\mathscr C^{\mathbf m}_{\text{local}}(b,c):\mathtt s\in(-\delta,\delta)\mapsto\Big(a_{\mathbf m}(\mathtt s,b,c)\,,\,\check r_{\mathbf m}(\mathtt s,b,c)\Big)\in\mathbb R\times X^{s,\sigma}_{\mathbf m},\qquad\delta>0\tag{2.55}$$
the corresponding (real-analytic) local curve which satisfies
$$a_{\mathbf m}(0,b,c)=a_{\mathbf m}(b,c),\qquad\frac{d}{d\mathtt s}\Big(\check r_{\mathbf m}(\mathtt s,b,c)\Big)\Big|_{\mathtt s=0}=\check r_{0,\mathbf m,b,c}.\tag{2.56}$$
► **Pitchfork-type bifurcation :** As in the subsection 2.1, we can write
$$d^2_{(\check r_+,\check r_-)}F\big(a_{\mathbf m}(b,c),b,c,0,0\big)[\check r_{0,\mathbf m,b,c},\check r_{0,\mathbf m,b,c}]=d_{(\check r_+,\check r_-)}F\big(a_{\mathbf m}(b,c),b,c,0,0\big)[\theta_{0,\mathbf m,b,c}],$$
with
$$\theta_{0,\mathbf m,b,c}(x)\triangleq2\pi\mathbf mM^{-1}_{2\mathbf m}\big(a_{\mathbf m}(b,c),b,c)\big)\begin{pmatrix}\Big(2\pi\mathbf m\big(a_{\mathbf m}(b,c)-c\big)-\frac{1}{2\pi\mathbf m\big(b-a_{\mathbf m}(b,c)\big)}\Big)^2\\ \frac{1}{4\pi^2\mathbf m^2\big(b-a_{\mathbf m}(b,c)\big)^2}\end{pmatrix}\cos(4\pi\mathbf mx).$$
Therefore, Theorem A.1 applies and gives
$$\frac{d}{d\mathtt s}\Big(a_{\mathbf m}(\mathtt s,b,c)\Big)\Big|_{\mathtt s=0}=0$$
and
$$\frac{d^2}{d\mathtt s^2}\Big(a_{\mathbf m}(\mathtt s,b,c)\Big)\Big|_{\mathtt s=0}=\frac{\Big\langle d^2_{(\check r_+,\check r_-)}F\big(a_{\mathbf m}(b,c),b,c,0,0\big)[\check r_{0,\mathbf m,b,c},\theta_{0,\mathbf m,b,c}]\,,\,g_{0,\mathbf m,b,c}\Big\rangle}{\big\langle\partial_a d_{(\check r_+,\check r_-)}F\big(a_{\mathbf m}(b,c),b,c,0,0\big)[\check r_{0,\mathbf m,b,c}]\,,\,g_{0,\mathbf m,b,c}\big\rangle}.\tag{2.57}$$
In addition,
$$\Big\langle d^2_{(\check r_+,\check r_-)}F\big(a_{\mathbf m}(b,c),b,c,0,0\big)[\check r_{0,\mathbf m,b,c},\theta_{0,\mathbf m,b,c}]\,,\,g_{0,\mathbf m,b,c}\Big\rangle=\frac{\pi^2\mathbf m^2}{3}\widetilde{\mathtt h}(\mathbf m,b,c),$$
where
$$\begin{aligned}\widetilde{\mathtt h}(\mathbf m,b,c)\triangleq{}&\alpha_{2\mathbf m}\big(a_{\mathbf m}(b,c),b,c\big)\gamma^4_{\mathbf m}\big(a_{\mathbf m}(b,c),b\big)-\beta_{2\mathbf m}\big(a_{\mathbf m}(b,c),b,c\big)\beta^4_{\mathbf m}\big(a_{\mathbf m}(b,c),b,c\big)\\&-2\gamma_{2\mathbf m}\big(a_{\mathbf m}(b,c),b\big)\gamma^2_{\mathbf m}\big(a_{\mathbf m}(b,c),b\big)\beta^2_{\mathbf m}\big(a_{\mathbf m}(b,c),b,c\big).\end{aligned}$$
After tedious calculations using (2.48), we can write
$$\widetilde{\mathtt h}(\mathbf m,b,c)=\frac{1}{64\pi^5\mathbf m^5(b-c)^5\big(4\pi^2\mathbf m^2(b-c)^2-1\big)^5}\Big(\mathtt h_1\big(4\pi^2\mathbf m^2(b-c)^2\big)+4\Big),$$
with $\mathtt h_1$ as in (2.46). Therefore, together with the constraint (2.49) and the variations of $\mathtt h_1$ obtained in the previous subsection, we deduce that
$$\widetilde{\mathtt h}(\mathbf m,a,c)>0.$$
This implies in turn
$$\Big\langle d^2_{(\check r_+,\check r_-)}F\big(a_{\mathbf m}(b,c),b,c,0,0\big)[\check r_{0,\mathbf m,b,c},\theta_{0,\mathbf m,b,c}]\,,\,g_{0,\mathbf m,b,c}\Big\rangle>0.\tag{2.58}$$
Combining (2.57), (2.53) and (2.58), we obtain
$$\begin{cases}\frac{d^2}{d\mathtt s^2}\Big(a_{\mathbf m}(\mathtt s,b,c)\Big)\Big|_{\mathtt s=0}>0, & \text{if } b>c,\\ \frac{d^2}{d\mathtt s^2}\Big(a_{\mathbf m}(\mathtt s,b,c)\Big)\Big|_{\mathtt s=0}<0, & \text{if } b<c.\end{cases}$$
This concludes the proof of Theorem 1.1-(iii).

2.4 Area bifurcation from symmetric flat strips

In this subsection, we look for solutions close to the symmetric flat strip $S_{\text{flat}}(-a,a)$. Hence the couple (a,b) is replaced by the couple $(-a,a)$ with $a>0$ and the functional F is replaced by

$$G:(0,\infty)\times\mathbb{R}\times X^{s,\sigma}_{\mathbf{m}}\to Y^{s-1,\sigma}_{\mathbf{m}},\qquad G(a,c,\check{r}_+,\check{r}_-)\triangleq F(-a,a,c,\check{r}_+,\check{r}_-).$$

In the sequel, refering to the notations (2.9) and (2.10), we denote

$$\widetilde{M_j}(a,c)\triangleq M_j(-a,a,c)=\begin{pmatrix}2\pi j(a-c)+\frac{1}{4\pi ja} & -\frac{1}{4\pi ja}\\ \frac{1}{4\pi ja} & -2\pi j(a+c)-\frac{1}{4\pi ja}\end{pmatrix}\triangleq\begin{pmatrix}\widetilde{\alpha}_j(a,c) & -\widetilde{\gamma}_j(a)\\ \widetilde{\gamma}_j(a) & -\widetilde{\beta}_j(a,c)\end{pmatrix},\tag{2.59}$$

$$\widetilde{\Delta}_j(a,c)\triangleq\Delta_j(-a,a,c)=-4\pi^2j^2(a^2-c^2)-1.\tag{2.60}$$

Notice that the velocity bifurcation is a consequence of Section 2.1 but the bifurcation from a differs from the previous studies. Remark that $2a$ corresponds to area of the flat strip $S_{\text{flat}}(-a,a)$. We fix $c\in\mathbb{R}$ and study the bifurcation with respect to the parameter $a>0$. Observe that for $c=0$, the relation (2.60) implies that for any $j\in\mathbb{N}^*$, the matrix $\widetilde{M_j}(a,0)$ is always invertible whatever the value of $a>0$. Hence, in the sequel, we shall restrict the discussion to the case $c\in\mathbb{R}^*$. According to (2.60), we have

$$\widetilde{\Delta}_j(a,c)=0\qquad\Leftrightarrow\qquad a=a_j(c)\triangleq\sqrt{\frac{4\pi^2j^2c^2-1}{4\pi^2j^2}}.\tag{2.61}$$

Notice that $a_j(c)$ is well-defined only for

$$j\geqslant N_1(c).$$

▶ **One dimensional kernel condition :** The sequence $\big(a_j(c)\big)_{j\geqslant N_1(c)}$ is positive, increasing and converges to $|c|$ when $j\to\infty$. In addition, for any fixed $\mathbf{m}\in\mathbb{N}^*$ with $\mathbf{m}\geqslant N_1(c)$, we have

$$\widetilde{\Delta}_{\mathbf{m}}\big(a_{\mathbf{m}}(c),c\big)=0$$

and

$$\forall j\in\mathbb{N}\setminus\{0,1\},\quad\widetilde{\Delta}_{j\mathbf{m}}\big(a_{\mathbf{m}}(c),c\big)=j^2-1>0.\tag{2.62}$$

Thus, the kernel of $d_{(\check{r}_+,\check{r}_-)}G\big(a_{\mathbf{m}}(c),c,0,0\big)$ is one dimensional, more precisely

$$\begin{aligned}&\ker\Big(d_{(\check{r}_+,\check{r}_-)}G\big(a_{\mathbf{m}}(c),c,0,0\big)\Big)=\mathtt{span}\big(\check{r}_{0,\mathbf{m},c}\big),\\ &\check{r}_{0,\mathbf{m},c}(x)\triangleq\begin{pmatrix}2\pi\mathbf{m}\big(a_{\mathbf{m}}(c)+c\big)+\frac{1}{4\pi\mathbf{m}a_{\mathbf{m}}(c)}\\ \frac{1}{4\pi\mathbf{m}a_{\mathbf{m}}(c)}\end{pmatrix}\cos(2\pi\mathbf{m}x).\end{aligned}\tag{2.63}$$

▶ **Range condition :** Notice that the monotonicity and the convergence of the sequence $\big(a_j(c)\big)_{j\geqslant N_1(c)}$ imply that $|c|\neq a_{\mathbf{m}}(c)$. So the Fredholmness property (2.7) gives that the range $\mathcal{R}\Big(d_{(\check{r}_+,\check{r}_-)}G\big(a_{\mathbf{m}}(c),c,0,0\big)\Big)$ is closed and of codimension one in $Y^{s-1,\sigma}_{\mathbf{m}}$. More precisely, arguying as in Section 2.1, we find

$$\begin{aligned}&\mathcal{R}\Big(d_{(\check{r}_+,\check{r}_-)}G\big(a_{\mathbf{m}}(c),c,0,0\big)\Big)=\Big(\mathtt{span}\big(g_{0,\mathbf{m},c}\big)\Big)^{\perp}=\ker\Big(g\mapsto\big\langle g,g_{0,\mathbf{m},c}\big\rangle\Big),\\ &g_{0,\mathbf{m},c}(x)\triangleq\begin{pmatrix}2\pi\mathbf{m}\big(a_{\mathbf{m}}(c)+c\big)+\frac{1}{4\pi\mathbf{m}a_{\mathbf{m}}(c)}\\ -\frac{1}{4\pi\mathbf{m}a_{\mathbf{m}}(c)}\end{pmatrix}\sin(2\pi\mathbf{m}x),\end{aligned}\tag{2.64}$$

where the orthogonal is understood in the sense of the scalar product defined in (2.17).

▶ **Transversality condition :** From (2.5)-(2.6), we have

$$\partial_a d_{(\check{r}_+,\check{r}_-)}G(a,c,0,0)=\begin{pmatrix}\partial_x & 0\\ 0 & -\partial_x\end{pmatrix}+\frac{1}{2a^2}\begin{pmatrix}\partial_x^{-1} & -\partial_x^{-1}\\ \partial_x^{-1} & -\partial_x^{-1}\end{pmatrix}.$$

Hence, direct calculations using in particular (1.24) give

$$\partial_a d_{(\check{r}_+,\check{r}_-)}G\big(a_{\mathbf{m}}(c),c,0,0\big)[\check{r}_{0,\mathbf{m},c}]=h_{0,\mathbf{m},c}-2\pi\mathbf{m}\,g_{0,\mathbf{m},c},$$

where

$$h_{0,\mathbf{m},c}(x)\triangleq\frac{a_{\mathbf{m}}(c)+c}{2a^2_{\mathbf{m}}(c)}\begin{pmatrix}1\\1\end{pmatrix}\sin(2\pi\mathbf{m}x).$$

Hence, one has the equivalence

$$\begin{aligned}&\partial_a d_{(\check r_+,\check r_-)}G\big(a_{\mathbf m}(c),c,0,0\big)[\check r_{0,\mathbf m,c}]\in\mathcal R\Big(d_{(\check r_+,\check r_-)}G\big(a_{\mathbf m}(c),c,0,0\big)\Big)\\ \Leftrightarrow\quad & 2\pi\mathbf m\big\langle g_{0,\mathbf m,c}\,,\,g_{0,\mathbf m,c}\big\rangle=\big\langle h_{0,\mathbf m,c}\,,\,g_{0,\mathbf m,c}\big\rangle.\end{aligned}\tag{2.65}$$

Now, on one hand, we have

$$\begin{aligned}\big\langle h_{0,\mathbf m,c}\,,\,g_{0,\mathbf m,c}\big\rangle&=\frac{a_{\mathbf m}(c)+c}{4a_{\mathbf m}^2(c)}\left\langle\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}2\pi\mathbf m\big(a_{\mathbf m}(c)+c\big)+\frac{1}{4\pi\mathbf m a_{\mathbf m}(c)}\\-\frac{1}{4\pi\mathbf m a_{\mathbf m}(c)}\end{pmatrix}\right\rangle_{\mathbb R^2}\\&=\frac{\pi\mathbf m\big(a_{\mathbf m}(c)+c\big)^2}{2a_{\mathbf m}^2(c)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}2\pi\mathbf m\big\langle g_{0,\mathbf m,c}\,,\,g_{0,\mathbf m,c}\big\rangle&=\pi\mathbf m\left[\left(2\pi\mathbf m\big(a_{\mathbf m}(c)+c\big)+\frac{1}{4\pi\mathbf m a_{\mathbf m}(c)}\right)^2+\left(\frac{1}{4\pi\mathbf m a_{\mathbf m}(c)}\right)^2\right]\\&=\frac{\pi\mathbf m}{a_{\mathbf m}^2(c)}\left[4\pi^2\mathbf m^2a_{\mathbf m}^2(c)\big(a_{\mathbf m}(c)+c\big)^2+a_{\mathbf m}(c)\big(a_{\mathbf m}(c)+c\big)+\frac{1}{8\pi^2\mathbf m^2}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}&\big\langle h_{0,\mathbf m,c}-2\pi\mathbf m g_{0,\mathbf m,c}\,,\,g_{0,\mathbf m,c}\big\rangle\\&=\frac{\pi\mathbf m}{a_{\mathbf m}^2(c)}\left[\frac12\big(a_{\mathbf m}(c)+c\big)^2-4\pi^2\mathbf m^2a_{\mathbf m}^2(c)\big(a_{\mathbf m}(c)+c\big)^2-a_{\mathbf m}(c)\big(a_{\mathbf m}(c)+c\big)-\frac{1}{8\pi^2\mathbf m^2}\right]\\&=-\frac{\pi\mathbf m}{a_{\mathbf m}^2(c)}\left[4\pi^2\mathbf m^2a_{\mathbf m}^2(c)\big(a_{\mathbf m}(c)+c\big)^2+\frac{a_{\mathbf m}^2(c)}{2}-\frac{c^2}{2}+\frac{1}{8\pi^2\mathbf m^2}\right].\end{aligned}\tag{2.66}$$

Now, the definition (2.61) implies

$$\frac{a_{\mathbf m}^2(c)}{2}-\frac{c^2}{2}+\frac{1}{8\pi^2\mathbf m^2}=0.$$

Plugging this identity into (2.66) and using the fact that $a_{\mathbf m}(c)\neq|c|$, yields

$$\begin{aligned}\big\langle\partial_a d_{(\check r_+,\check r_-)}G\big(a_{\mathbf m}(c),c,0,0\big)[\check r_{0,\mathbf m,c}]\,,\,g_{0,\mathbf m,c}\big\rangle&=\big\langle h_{0,\mathbf m,c}-2\pi\mathbf m\,g_{0,\mathbf m,c}\,,\,g_{0,\mathbf m,c}\big\rangle\\&=-4\pi^3\mathbf m^3\big(c+a_{\mathbf m}(c)\big)^2<0.\end{aligned}\tag{2.67}$$

Consequently, the relation (2.65) is not satisfied, that is

$$\partial_a d_{(\check r_+,\check r_-)}G\big(a_{\mathbf m}(c),c,0,0\big)[\check r_{0,\mathbf m,c}]\not\in\mathcal R\Big(d_{(\check r_+,\check r_-)}G\big(a_{\mathbf m}(c),c,0,0\big)\Big).\tag{2.68}$$

Finally, (2.2), (2.4), (2.7), (2.63) and (2.68) allow to apply the Crandall-Rabinowitz Theorem A.1 which concludes the existence part of Theorem 1.1-(iv). In the sequel, we denote

$$\mathscr C_{\textrm{local}}^{\mathbf m}(c):\mathtt s\in(-\delta,\delta)\mapsto\Big(a_{\mathbf m}(\mathtt s,c)\,,\,\check r_{\mathbf m}(\mathtt s,c)\Big)\in\mathbb T\times X_{\mathbf m}^{s,\sigma},\qquad\delta>0$$

the corresponding (real-analytic) local curve which satisfies

$$a_{\mathbf m}(0,c)=a_{\mathbf m}(c),\qquad\frac{d}{d\mathtt s}\Big(\check r_{\mathbf m}(\mathtt s,c)\Big)\Big|_{\mathtt s=0}=\check r_{0,\mathbf m,c}.$$

► **Pitchfork-type bifurcation :** Proceeding as in subsection 2.1, we can write

$$d_{(\check r_+,\check r_-)}G\big(a_{\mathbf m}(c),c,0,0\big)[\theta_{0,\mathbf m,c}]=d^2_{(\check r_+,\check r_-)}G\big(a_{\mathbf m}(c),c,0,0\big)[\check r_{0,\mathbf m,c},\check r_{0,\mathbf m,c}],$$

with

$$\theta_{0,\mathbf m,c}(x)\triangleq2\pi\mathbf m\widetilde M_{2\mathbf m}^{-1}\big(a_{\mathbf m}(c),c\big)\begin{pmatrix}\Big(2\pi\mathbf m\big(c+a_{\mathbf m}(c)\big)+\frac{1}{4a_{\mathbf m}(c)\pi\mathbf m}\Big)^2\\\frac{1}{16a_{\mathbf m}^2(c)\pi^2\mathbf m^2}\end{pmatrix}\cos(4\pi\mathbf m x).$$

Then, according to Theorem A.1, and (2.26), we have

$$\frac{d}{d\mathtt s}\Big(a_{\mathbf m}(\mathtt s,c)\Big)\Big|_{\mathtt s=0}=0$$

and

$$\frac{d^2}{d\mathtt{s}^2}\Big(a_{\mathbf{m}}(\mathtt{s},c)\Big)\Big|_{\mathtt{s}=0}=\frac{\Big\langle d^2_{(\check{r}_+,\check{r}_-)}G\big(a_{\mathbf{m}}(c),c,0,0\big)[\check{r}_{0,\mathbf{m},c},\theta_{0,\mathbf{m},c}]\,,\,g_{0,\mathbf{m},c}\Big\rangle}{\big\langle\partial_c d_{(\check{r}_+,\check{r}_-)}G\big(a_{\mathbf{m}}(c),c,0,0\big)[\check{r}_{0,\mathbf{m},c}]\,,\,g_{0,\mathbf{m},c}\big\rangle}\cdot \tag{2.69}$$

Proceeding as in the subsection 2.1, we obtain

$$\Big\langle d^2_{(\check{r}_+,\check{r}_-)}G\big(a_{\mathbf{m}}(c),c,0,0\big)[\check{r}_{0,\mathbf{m},c},\theta_{0,\mathbf{m},c}]\,,\,g_{0,\mathbf{m},c}\Big\rangle=\frac{\pi^2\mathbf{m}^2}{3}\mathtt{f}(\mathbf{m},c), \tag{2.70}$$

where $\mathtt{f}$ is a well-defined continuous fonction on $D_{\mathtt{f}}\triangleq\{(z,c)\in[1,\infty)\times\mathbb{R}^*\quad\text{s.t.}\quad z\geqslant N_1(c)\}$ given by

$$\mathtt{f}(z,c)\triangleq\Big[\widetilde{\alpha}_{2z}\big(a_z(c),c\big)\widetilde{\gamma}_z^4\big(a_z(c)\big)+\widetilde{\beta}_{2z}\big(a_z(c),c\big)\widetilde{\beta}_z^4\big(a_z(c),c\big)-2\widetilde{\gamma}_{2z}\big(a_z(c)\big)\widetilde{\gamma}_z^2\big(a_z(c)\big)\widetilde{\beta}_z^2\big(a_z(c),c\big)\Big].$$

We warn the reader about the change of notation for the coefficients, which explains the different shape of $\mathtt{f}$ compared to the other subsections. After tedious calculations, we can write

$$\mathtt{f}(z,c)=\frac{1}{128\pi^5z^5a_z^5(c)}\Big(C(z,c)a_z(c)+D(z,c)\Big), \tag{2.71}$$

where

$$\begin{aligned}C(z,c)&\triangleq 131072\pi^{10}z^{10}c^9-71680\pi^8z^8c^7+13056\pi^6z^6c^5-896\pi^4z^4c^3+16\pi^2z^2c,\\ D(z,c)&\triangleq 131072\pi^{10}z^{10}c^{10}-88064\pi^8z^8c^8+20992\pi^6z^6c^6-2096\pi^4z^4c^4+80\pi^2z^2c^2-1.\end{aligned}$$

Assume, for the sake of contradiction, that there exists $(z,c)\in D_{\mathtt{f}}$ such that $\mathtt{f}(z,c)=0$. Then this equation is equivalent to

$$a_z(c)=-\frac{D(z,a)}{C(z,a)}\cdot$$

Taking the square of the previous expression and using (2.61), we end up with

$$1048576\pi^{12}z^{12}c^{12}+284928\pi^8z^8c^8+3168\pi^4z^4c^4+1=884736\pi^{10}z^{10}c^{10}+43520\pi^6z^6c^6+96\pi^2z^2c^2.$$

Now by construction $z\geqslant N_1(c)$, which implies in particular $2\pi|c|z\geqslant 1$. Consequently, the previous equation cannot be satisfied and we obtain

$$\forall(z,c)\in D_{\mathtt{f}},\quad \mathtt{f}(z,c)\neq 0.$$

Besides, the convergence of $a_z(c)$ to $|c|$ when z is large together with (2.71) provide the following asymptotics

$$\forall c>0,\quad \mathtt{f}(z,c)\underset{z\to\infty}{\sim}2048\pi^5z^5c^5\qquad\text{and}\qquad\forall c<0,\quad \mathtt{f}(z,c)\underset{z\to\infty}{\sim}-128\pi^3z^3|c|^3.$$

Therefore, by a continuity argument, we obtain

$$\forall(z,c)\in D_{\mathtt{f}},\quad\begin{cases}\mathtt{f}(z,c)>0, & \text{if } c>0,\\ \mathtt{f}(z,c)<0, & \text{if } c<0.\end{cases}$$

Added to (2.70) and (2.62), we obtain

$$\begin{cases}\Big\langle d^2_{(\check{r}_+,\check{r}_-)}G\big(a_{\mathbf{m}}(c),c,0,0\big)[\check{r}_{0,\mathbf{m},c},\theta_{0,\mathbf{m},c}]\,,\,g_{0,\mathbf{m},c}\Big\rangle>0, & \text{if } c>0,\\ \Big\langle d^2_{(\check{r}_+,\check{r}_-)}G\big(a_{\mathbf{m}}(c),c,0,0\big)[\check{r}_{0,\mathbf{m},c},\theta_{0,\mathbf{m},c}]\,,\,g_{0,\mathbf{m},c}\Big\rangle<0, & \text{if } c<0.\end{cases} \tag{2.72}$$

Plugging (2.67) and (2.72) into (2.69) yields

$$\forall c>0,\quad\frac{d^2}{d\mathtt{s}^2}\Big(a_{\mathbf{m}}(\mathtt{s},c)\Big)\Big|_{\mathtt{s}=0}<0\qquad\text{and}\qquad\forall c<0,\quad\frac{d^2}{d\mathtt{s}^2}\Big(a_{\mathbf{m}}(\mathtt{s},c)\Big)\Big|_{\mathtt{s}=0}>0.$$

This ends the proof of Theorem 1.1-(iv).

3 Large amplitude solutions

The scope of this section is to implement the analytic global bifurcation Theorem A.2 in order to continue the local branches constructed in the previous section. We first discuss some qualitative properties for a solution and then implement the global bifurcation theory.

3.1 Qualitative properties of generic solutions

Let us fix $(a,b,c)\in\mathbb{S}\times\mathbb{R}$ and consider $(\check{r}_+,\check{r}_-)$ a real-analytic non-trivial solution to the system (2.1). Notice that, in order to have an electron layer, we must have

$$\forall(t,x)\in\mathbb{R}_+\times\mathbb{T},\quad v_+(t,x)>v_-(t,x),$$

which is equivalent to

$$\forall x\in\mathbb{T},\quad \check{r}_+(x)+b-c>\check{r}_-(x)+a-c. \tag{3.1}$$

Substracting both equations in (2.1) yields

$$\big(\check{r}_-(x)+a-c\big)\partial_x\check{r}_-(x)=\big(\check{r}_+(x)+b-c\big)\partial_x\check{r}_+(x).$$

From the previous two relations, we deduce that for any $\overline{x},\underline{x}\in\mathbb{T}$, we have

$$\begin{aligned}\check{r}_+(\overline{x})+b-c=0\quad&\Rightarrow\quad\Big(\check{r}_-(\overline{x})+a-c<0\quad\text{and}\quad\partial_x\check{r}_-(\overline{x})=0\Big),\\ \check{r}_-(\underline{x})+a-c=0\quad&\Rightarrow\quad\Big(\check{r}_+(\underline{x})+b-c>0\quad\text{and}\quad\partial_x\check{r}_+(\underline{x})=0\Big).\end{aligned}$$

Let us assume that there exists $\overline{x}\in\mathbb{T}$ such that $\check{r}_+(\overline{x})+b-c=0$ and consider the following integers (finite by real-analyticity and zero average condition)

$$\begin{aligned}\overline{q}_0^+&\triangleq\min\big\{q\in\mathbb{N}^*\quad\text{s.t.}\quad\partial_x^q\check{r}_+(\overline{x})\neq0\big\}\in\mathbb{N}^*,\\ \overline{q}_0^-&\triangleq\min\big\{q\in\mathbb{N}^*\quad\text{s.t.}\quad\partial_x^q\check{r}_-(\overline{x})\neq0\big\}\in\mathbb{N}\setminus\{0,1\}.\end{aligned}$$

Using Leibniz rule, we obtain for any $q\in\mathbb{N}$,

$$\begin{cases}\big(\check{r}_+(x)+b-c\big)\partial_x^{q+1}\check{r}_+(x)+\displaystyle\sum_{k=1}^{q}\binom{q}{k}\partial_x^k\check{r}_+(x)\partial_x^{q-k+1}\check{r}_+(x)-\tfrac{1}{b-a}\partial_x^{q-1}\big(\check{r}_+(x)-\check{r}_-(x)\big)=0,\\ \big(\check{r}_-(x)+a-c\big)\partial_x^{q+1}\check{r}_-(x)+\displaystyle\sum_{k=1}^{q}\binom{q}{k}\partial_x^k\check{r}_-(x)\partial_x^{q-k+1}\check{r}_-(x)-\tfrac{1}{b-a}\partial_x^{q-1}\big(\check{r}_+(x)-\check{r}_-(x)\big)=0.\end{cases} \tag{3.2}$$

Substracting both equations in (3.2), we obtain

$$\forall q\in\mathbb{N}^*,\quad\partial_x^{q+1}\check{r}_-(\overline{x})=\frac{1}{\check{r}_-(\overline{x})+a-c}\sum_{k=1}^{q}\binom{q}{k}\Big[\partial_x^k\check{r}_+(\overline{x})\partial_x^{q-k+1}\check{r}_+(\overline{x})-\partial_x^k\check{r}_-(\overline{x})\partial_x^{q-k+1}\check{r}_-(\overline{x})\Big].$$

From this, we readily infer

$$\overline{q}_0^-=\begin{cases}\overline{q}_0^++1,&\text{if }\overline{q}_0^+\equiv1[2],\\ \overline{q}_0^++2,&\text{if }\overline{q}_0^+\equiv0[2].\end{cases}$$

By a similar argument, for any $\underline{x}\in\mathbb{T}$ with $\check{r}_-(\underline{x})+a-c=0$, then denoting

$$\begin{aligned}\underline{q}_0^+&\triangleq\min\big\{q\in\mathbb{N}^*\quad\text{s.t.}\quad\partial_x^q\check{r}_+(\underline{x})\neq0\big\}\in\mathbb{N}\setminus\{0,1\},\\ \underline{q}_0^-&\triangleq\min\big\{q\in\mathbb{N}^*\quad\text{s.t.}\quad\partial_x^q\check{r}_-(\underline{x})\neq0\big\}\in\mathbb{N}^*,\end{aligned}$$

we have

$$\underline{q}_0^+=\begin{cases}\underline{q}_0^-+1,&\text{if }\underline{q}_0^-\equiv1[2],\\ \underline{q}_0^-+2,&\text{if }\underline{q}_0^-\equiv0[2].\end{cases}$$

3.2 Global bifurcation

Now, we prove the Theorem 1.2. We denote

$$\mathtt{m}(a,b)\triangleq\min_{x\in\mathbb{T}}\big|\check{r}_+(x)-\check{r}_-(x)+b-a\big|,\qquad\mathtt{m}_+(b,c)\triangleq\min_{x\in\mathbb{T}}\big|\check{r}_+(x)+b-c\big|,\qquad\mathtt{m}_-(a,c)\triangleq\min_{x\in\mathbb{T}}\big|\check{r}_-(x)+a-c\big|.$$

Due to the similarity of the argument, we shall mainly focus on proving the global bifurcation from the velocity parameter c when the other parameters a and b are fixed. We introduce the following open set

$$U(a,b)\triangleq\Big\{(c,\check{r}_+,\check{r}_-)\in\mathbb{R}\times X_{\mathbf{m}}^{s,\sigma}\quad\text{s.t.}\quad\min\big(\mathtt{m}(a,b),\mathtt{m}_+(b,c),\mathtt{m}_-(a,c)\big)>0\Big\}.$$

In the sequel, we shall denote for any $r>0$

$$B_{\mathbf{m}}^{s,\sigma}(r) \triangleq \Big\{ f\in X_{\mathbf{m}}^{s,\sigma} \quad \text{s.t.} \quad \|f\|_{s,\sigma}\leqslant r\Big\}.$$

Let us consider the following closed and bounded set defined for any $n\in\mathbb{N}^*$ by

$$F_n(a,b) \triangleq \Big\{ (c,\check{r}_+,\check{r}_-)\in[-n,n]\times B_{\mathbf{m}}^{s,\sigma}(n) \quad \text{s.t.} \quad \min\big(\mathtt{m}(a,b),\mathtt{m}_+(b,c),\mathtt{m}_-(a,c)\big)\geqslant \tfrac{1}{n}\Big\}.$$

Obviously, one has

$$U(a,b)=\bigcup_{n\in\mathbb{N}^*}F_n(a,b).$$

We denote

$$\mathscr{S}_n(a,b) \triangleq \Big\{ (c,\check{r}_+,\check{r}_-)\in F_n(a,b) \quad \text{s.t.} \quad F(a,b,c,\check{r}_+,\check{r}_-)=0\Big\}.$$

Lemma 3.1. *Let $a<b$, $s>\frac{3}{2}$ and $\sigma>0$. The following properties hold true.*

(i) We have the inclusion $\mathscr{C}_{\text{local}}^{\pm,\mathbf{m}}(a,b)\subset U(a,b)$.

(ii) For any $(c,\check{r}_+,\check{r}_-)\in U(a,b)$ with $F(a,b,c,\check{r}_+,\check{r}_-)=0$, the operator $d_{(\check{r}_+,\check{r}_-)}F(a,b,c,\check{r}_+,\check{r}_-):X_{\mathbf{m}}^{s,\sigma}\to Y_{\mathbf{m}}^{s-1,\sigma}$ is Fredholm with index zero.

(iii) For any $n\in\mathbb{N}^*$, the set $\mathscr{S}_n(a,b)$ is compact in $\mathbb{R}\times X_{\mathbf{m}}^{s,\sigma}$.

Proof. **(i)** Let $\kappa\in\{+,-\}$. Since $a<b$ and $c_{\mathbf{m}}^{\kappa}(a,b)\notin\{a,b\}$, then, up to taking δ small enough we get

$$\Big|\big(\check{r}_{\mathbf{m}}^{\kappa}(\mathtt{s},a,b)\big)_+(x)-\big(\check{r}_{\mathbf{m}}^{\kappa}(\mathtt{s},a,b)\big)_-(x)+b-a\Big|\geqslant|b-a|-C\delta>0,$$
$$\Big|\big(\check{r}_{\mathbf{m}}^{\kappa}(\mathtt{s},a,b)\big)_+(x)+b-c_{\mathbf{m}}^{\kappa}(\mathtt{s},a,b)\Big|\geqslant|b-c_{\mathbf{m}}^{\kappa}(a,b)|-C\delta>0,$$
$$\Big|\big(\check{r}_{\mathbf{m}}^{\kappa}(\mathtt{s},a,b)\big)_-(x)+a-c_{\mathbf{m}}^{\kappa}(\mathtt{s},a,b)\Big|\geqslant|a-c_{\mathbf{m}}^{\kappa}(a,b)|-C\delta>0.$$

This proves the inclusion $\mathscr{C}_{\text{local}}^{\kappa,\mathbf{m}}(a,b)\subset U(a,b)$.
(ii) Let $(c,\check{r}_+,\check{r}_-)\in U(a,b)$ with $F(a,b,c,\check{r}_+,\check{r}_-)=0$. Differentiating (2.1), we can write

$$d_{(\check{r}_+,\check{r}_-)}F(a,b,c,\check{r}_+,\check{r}_-)=I_{(\check{r}_+,\check{r}_-)}+K_{(\check{r}_+,\check{r}_-)},$$

where

$$I_{(\check{r}_+,\check{r}_-)}\triangleq\begin{pmatrix}(\check{r}_++b-c)\partial_x & 0\\ 0 & (\check{r}_-+a-c)\partial_x\end{pmatrix}$$

and

$$K_{(\check{r}_+,\check{r}_-)}\triangleq M_{(\check{r}_+,\check{r}_-)}+K_{(0,0)},\qquad M_{(\check{r}_+,\check{r}_-)}\triangleq\begin{pmatrix}\partial_x\check{r}_+ & 0\\ 0 & \partial_x\check{r}_-\end{pmatrix}.$$

Since $(c,\check{r}_+,\check{r}_-)\in U(a,b)$, then in particular

$$\forall x\in\mathbb{T},\quad \check{r}_+(x)+b-c\neq0\qquad\text{and}\qquad\check{r}_-(x)+a-c\neq0.$$

This implies that the operator $I_{(\check{r}_+,\check{r}_-)}:X_{\mathbf{m}}^{s,\sigma}\to Y_{\mathbf{m}}^{s-1,\sigma}$ is an isomorphism. Now, recall that the compactness of $K_{(0,0)}:X_{\mathbf{m}}^{s,\sigma}\to Y_{\mathbf{m}}^{s-1,\sigma}$ has already been proved at the beginning of Section 2. Let us now prove the compactness of $M_{(\check{r}_+,\check{r}_-)}:X_{\mathbf{m}}^{s,\sigma}\to Y_{\mathbf{m}}^{s-1,\sigma}$. For this aim, consider $\left(x_+^{[k]},x_-^{[k]}\right)_{k\in\mathbb{N}}$ bounded in $X_{\mathbf{m}}^{s,\sigma}$. There exist $(k_m)_{m\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}$ increasing and $\left(x_+^{[\infty]},x_-^{[\infty]}\right)\in X_{\mathbf{m}}^{s-1,\sigma}$ such that

$$\left(x_+^{[k_m]},x_-^{[k_m]}\right)\underset{m\to\infty}{\longrightarrow}\left(x_+^{[\infty]},x_-^{[\infty]}\right)\quad\text{in } X_{\mathbf{m}}^{s-1,\sigma}.$$

We denote, for any $m\in\mathbb{N}$,

$$\left(y_+^{[m]},y_-^{[m]}\right)\triangleq M_{(\check{r}_+,\check{r}_-)}\left(x_+^{[k_m]},x_-^{[k_m]}\right)=\left(\partial_x\check{r}_+\,x_+^{[k_m]},\partial_x\check{r}_-\,x_-^{[k_m]}\right)\in Y_{\mathbf{m}}^{s-1,\sigma}.$$

Our purpose is to prove the convergence of the sequence $\left(y_+^{[m]},y_-^{[m]}\right)_{m\in\mathbb{N}}$ in $Y_{\mathbf{m}}^{s-1,\sigma}$. Let $p,q\in\mathbb{N}$, then using that $s>\frac{3}{2}$, we have

$$\begin{aligned}\left\|y_\pm^{[p]}-y_\pm^{[q]}\right\|_{s-1,\sigma}&=\left\|\partial_x r_\pm\left(x_\pm^{[p]}-x_\pm^{[q]}\right)\right\|_{s-1,\sigma}\\&\lesssim\|\check{r}_\pm\|_{s,\sigma}\left\|x_\pm^{[k_p]}-x_\pm^{[k_q]}\right\|_{s-1,\sigma}.\end{aligned}$$

Since the sequence $\left(x_+^{[k_m]},x_-^{[k_m]}\right)_{m\in\mathbb{N}}$ is convergent in $X_{\mathbf{m}}^{s-1,\sigma}$, then it is of Cauchy-type in $X_{\mathbf{m}}^{s-1,\sigma}$. We deduce that the sequence $\left(y_+^{[m]},y_-^{[m]}\right)_{m\in\mathbb{N}}$ is of Cauchy-type (thus convergent) in the Banach space $Y_{\mathbf{m}}^{s-1,\sigma}$. Consequently, the operator $d_{(\check{r}_+,\check{r}_-)}F(a,b,c,\check{r}_+,\check{r}_-):X_{\mathbf{m}}^{s,\sigma}\to Y_{\mathbf{m}}^{s-1,\sigma}$ is a compact perturbation of an isomorphism. Therefore, it is a Fredholm operator with index zero.

(iii) Let $\left(c^{[k]},\check{r}_+^{[k]},\check{r}_-^{[k]}\right)_{k\in\mathbb{N}}\in\big(\mathscr{S}_n(a,b)\big)^{\mathbb{N}}$. Since $\left(c^{[k]}\right)_{k\in\mathbb{N}}\in[-n,n]^{\mathbb{N}}$, then by Bolzano-Weierstrass Theorem, there exists $c^{[\infty]}\in[-n,n]$ and a subsequence $(k_m)_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty}c^{[k_m]}=c^{[\infty]}.$$

Since $\left(\check{r}_+^{[k]},\check{r}_-^{[k]}\right)_{k\in\mathbb{N}}$ is bounded in $X_{\mathbf{m}}^{s,\sigma}$, then there exists $\left(\check{r}_+^{[\infty]},\check{r}_-^{[\infty]}\right)\in X_{\mathbf{m}}^{s,\sigma}$ such that, up to an other extraction,

$$\left(\check{r}_+^{[k_m]},\check{r}_-^{[k_m]}\right)\underset{m\to\infty}{\rightharpoonup}\left(\check{r}_+^{[\infty]},\check{r}_-^{[\infty]}\right)\quad\text{in }X_{\mathbf{m}}^{s,\sigma}$$

and

$$\forall\,\tfrac{3}{2}<s'<s,\quad\left(\check{r}_+^{[k_m]},\check{r}_-^{[k_m]}\right)\underset{m\to\infty}{\longrightarrow}\left(\check{r}_+^{[\infty]},\check{r}_-^{[\infty]}\right)\quad\text{in }X_{\mathbf{m}}^{s',\sigma}.$$

Since, $s'>\frac{3}{2}$, by pointwise convergence, we obtain

$$F\left(a,b,c^{[\infty]},\check{r}_+^{[\infty]},\check{r}_-^{[\infty]}\right)=0.$$

We shall now prove that the sequence $\left(\check{r}_+^{[k_m]},\check{r}_-^{[k_m]}\right)_{m\in\mathbb{N}}$ is a Cauchy sequence (and thus convergent) in the Banach space $X_{\mathbf{m}}^{s,\sigma}$. Let $p,q\in\mathbb{N}$, since

$$F\left(a,b,c^{[k_p]},\check{r}_+^{[k_p]},\check{r}_-^{[k_p]}\right)=0=F\left(a,b,c^{[k_q]},\check{r}_+^{[k_q]},\check{r}_-^{[k_q]}\right),$$

then substracting, we obtain

$$\partial_x\left(\check{r}_+^{[k_p]}-\check{r}_+^{[k_q]}\right)=\mathcal{I}_1^{+,p,q}+\mathcal{I}_2^{+,p,q},\qquad\partial_x\left(\check{r}_-^{[k_p]}-\check{r}_-^{[k_q]}\right)=\mathcal{I}_1^{-,p,q}+\mathcal{I}_2^{-,p,q},$$

where

$$\mathcal{I}_1^{+,p,q}\triangleq\frac{\left(c^{[k_p]}-c^{[k_q]}+\check{r}_+^{[k_q]}-\check{r}_+^{[k_p]}\right)\partial_x\check{r}_+^{[k_q]}}{r_+^{[k_p]}+b-c^{[k_p]}},\quad\mathcal{I}_1^{-,p,q}\triangleq\frac{\left(c^{[k_p]}-c^{[k_q]}+\check{r}_-^{[k_q]}-\check{r}_-^{[k_p]}\right)\partial_x\check{r}_-^{[k_q]}}{r_-^{[k_p]}+a-c^{[k_p]}},$$

$$\mathcal{I}_2^{+,p,q}\triangleq\frac{\partial_x^{-1}\left(\check{r}_+^{[k_p]}-\check{r}_+^{[k_q]}-\check{r}_-^{[k_p]}+\check{r}_-^{[k_q]}\right)}{(b-a)\left(r_+^{[k_p]}+b-c^{[k_p]}\right)},\qquad\mathcal{I}_2^{-,p,q}\triangleq\frac{\partial_x^{-1}\left(\check{r}_+^{[k_p]}-\check{r}_+^{[k_q]}-\check{r}_-^{[k_p]}+\check{r}_-^{[k_q]}\right)}{(b-a)\left(r_+^{[k_p]}+a-c^{[k_p]}\right)}.$$

Since

$$\forall p\in\mathbb{N},\quad\min\left(\min_{x\in\mathbb{T}}\left|\check{r}_+^{[k_p]}(x)+b-c^{[k_p]}\right|,\min_{x\in\mathbb{T}}\left|\check{r}_-^{[k_p]}(x)+a-c^{[k_p]}\right|\right)\geqslant\tfrac{1}{n},$$

then

$$\begin{aligned}\left\|\mathcal{I}_1^{\pm,p,q}\right\|_{s-1,\sigma}&\lesssim_{n,a,b}\left(\left|c^{[k_p]}-c^{[k_q]}\right|+\left\|\check{r}_\pm^{[k_p]}-\check{r}_\pm^{[k_q]}\right\|_{s-1,\sigma}\right)\left\|\partial_x\check{r}_\pm^{[k_q]}\right\|_{s-1,\sigma}\\&\lesssim_{n,a,b}\left(\left|c^{[k_p]}-c^{[k_q]}\right|+\left\|\check{r}_\pm^{[k_p]}-\check{r}_\pm^{[k_q]}\right\|_{s',\sigma}\right)\left\|\check{r}_\pm^{[k_q]}\right\|_{s,\sigma}\\&\lesssim_{n,a,b}\left(\left|c^{[k_p]}-c^{[k_q]}\right|+\left\|\check{r}_\pm^{[k_p]}-\check{r}_\pm^{[k_q]}\right\|_{s',\sigma}\right)\end{aligned}$$

and

$$\begin{aligned}\left\|\mathcal{I}_2^{\pm,p,q}\right\|_{s-1,\sigma}&\lesssim_{n,a,b}\left\|\partial_x^{-1}\left(\check{r}_+^{[k_p]}-\check{r}_+^{[k_q]}+\check{r}_-^{[k_p]}-\check{r}_-^{[k_q]}\right)\right\|_{s-1,\sigma}\\&\lesssim_{n,a,b}\left(\left\|\check{r}_+^{[k_p]}-\check{r}_+^{[k_q]}\right\|_{s',\sigma}+\left\|\check{r}_-^{[k_p]}-\check{r}_-^{[k_q]}\right\|_{s',\sigma}\right).\end{aligned}$$

Since the sequences $\left(c^{[k_m]}\right)_{m\in\mathbb{N}}$ and $\left(\check{r}_+^{[k_m]},\check{r}_-^{[k_m]}\right)$ and convergent in $\mathbb{R}$ are $X_{\mathbf{m}}^{s',\sigma}$ respectively, they are in particular of Cauchy-type in the corresponding spaces. This gives the desired result, i.e.

$$\left(\check{r}_+^{[k_m]},\check{r}_-^{[k_m]}\right)\underset{m\to\infty}{\longrightarrow}\left(\check{r}_+^{[\infty]},\check{r}_-^{[\infty]}\right)\quad\text{in }X_{\mathbf{m}}^{s,\sigma}.$$

Thus, for any $n\in\mathbb{N}^*$, the set $\mathscr{S}_n(a,b)$ is compact in $\mathbb{R}\times X_{\mathbf{m}}^{s,\sigma}$. This ends the proof of Lemma 3.1. ☐

Now, we can conclude.

Proof of Theorem 1.2-(i). Let $\kappa\in\{+,-\}$. The Lemma 3.1 allows to apply the Theorem A.2 which provides the existence of a global continuation curve $\mathscr{C}^{\kappa,\mathbf{m}}_{\text{global}}(a,b)$ satisfying

$$\mathscr{C}^{\kappa,\mathbf{m}}_{\text{local}}(a,b)\subset\mathscr{C}^{\kappa,\mathbf{m}}_{\text{global}}(a,b)\triangleq\Big\{\big(c^{\kappa}_{\mathbf{m}}(\mathtt{s},a,b),\check{r}^{\kappa}_{\mathbf{m}}(\mathtt{s},a,b)\big),\quad \mathtt{s}\in\mathbb{R}\Big\}\subset U(a,b)\cap F(a,b,\cdot,\cdot,\cdot)^{-1}\big(\{0\}\big).$$

Moreover, the curve $\mathscr{C}^{\kappa,\mathbf{m}}_{\text{global}}(a,b)$ admits locally around each of its points a real-analytic reparametrization. In addition, one of the following alternatives occurs

$(A1)$ There exists $T^{\kappa}_{\mathbf{m}}(a,b)>0$ such that

$$\forall\mathtt{s}\in\mathbb{R},\quad c^{\kappa}_{\mathbf{m}}\big(\mathtt{s}+T^{\kappa}_{\mathbf{m}}(a,b),a,b\big)=c^{\kappa}_{\mathbf{m}}(\mathtt{s},a,b)\qquad\text{and}\qquad \check{r}^{\pm}_{\mathbf{m}}\big(\mathtt{s}+T^{\kappa}_{\mathbf{m}}(a,b),a,b\big)=\check{r}^{\pm}_{\mathbf{m}}(\mathtt{s},a,b).$$

$(A2)$ One one the following limits holds (possibly simultaneously)

- (Blow-up) $\displaystyle\lim_{\mathtt{s}\to\pm\infty}\frac{1}{1+|c^{\kappa}_{\mathbf{m}}(\mathtt{s},a,b)|+\|\check{r}^{\kappa}_{\mathbf{m}}(\mathtt{s},a,b)\|_{s,\sigma}}=0.$
- (Collision of the boundaries) $\displaystyle\lim_{\mathtt{s}\to\pm\infty}\min_{x\in\mathbb{T}}\Big|\big(\check{r}^{\kappa}_{\mathbf{m}}\big)_{+}(\mathtt{s},a,b)(x)-\big(\check{r}^{\kappa}_{\mathbf{m}}\big)_{-}(\mathtt{s},a,b)(x)+b-a\Big|=0.$
- (Degeneracy $+$) $\displaystyle\lim_{\mathtt{s}\to\pm\infty}\min_{x\in\mathbb{T}}\Big|\big(\check{r}^{\kappa}_{\mathbf{m}}\big)_{+}(\mathtt{s},a,b)(x)+b-c^{\kappa}_{\mathbf{m}}(\mathtt{s},a,b)\Big|=0.$
- (Degeneracy $-$) $\displaystyle\lim_{\mathtt{s}\to\pm\infty}\min_{x\in\mathbb{T}}\Big|\big(\check{r}^{\kappa}_{\mathbf{m}}\big)_{-}(\mathtt{s},a,b)(x)+a-c^{\kappa}_{\mathbf{m}}(\mathtt{s},a,b)\Big|=0.$

This gives the desired result. □

Now, to prove the other items of Theorem 1.2, we proceed similarly by replacing $U(a,b)$ by one of the following open sets

$$V(a,c)\triangleq\Big\{(b,\check{r}_+,\check{r}_-)\in(a,\infty)\times X^{s,\sigma}_{\mathbf{m}}\quad\text{s.t.}\quad\min\big(\mathtt{m}(a,b),\mathtt{m}_+(b,c),\mathtt{m}_-(a,c)\big)>0\Big\},$$
$$W(b,c)\triangleq\Big\{(a,\check{r}_+,\check{r}_-)\in(-\infty,b)\times X^{s,\sigma}_{\mathbf{m}}\quad\text{s.t.}\quad\min\big(\mathtt{m}(a,b),\mathtt{m}_+(b,c),\mathtt{m}_-(a,c)\big)>0\Big\},$$
$$Z(c)\triangleq\Big\{(a,\check{r}_+,\check{r}_-)\in(0,\infty)\times X^{s,\sigma}_{\mathbf{m}}\quad\text{s.t.}\quad\min\big(\mathtt{m}(-a,a),\mathtt{m}_+(a,c),\mathtt{m}_-(-a,c)\big)>0\Big\}$$

and $F_n(a,b)$ by one of the following closed and bounded sets

$$G_n(a,c)\triangleq\Big\{(b,\check{r}_+,\check{r}_-)\in[a+\tfrac{1}{n},a+n]\times B^{s,\sigma}_{\mathbf{m}}(n)\quad\text{s.t.}\quad\min\big(\mathtt{m}(a,b),\mathtt{m}_+(b,c),\mathtt{m}_-(a,c)\big)\geqslant\tfrac{1}{n}\Big\},$$
$$H_n(b,c)\triangleq\Big\{(a,\check{r}_+,\check{r}_-)\in[b-n,b-\tfrac{1}{n}]\times B^{s,\sigma}_{\mathbf{m}}(n)\quad\text{s.t.}\quad\min\big(\mathtt{m}(a,b),\mathtt{m}_+(b,c),\mathtt{m}_-(a,c)\big)\geqslant\tfrac{1}{n}\Big\},$$
$$I_n(c)\triangleq\Big\{(a,\check{r}_+,\check{r}_-)\in[\tfrac{1}{n},n]\times B^{s,\sigma}_{\mathbf{m}}(n)\quad\text{s.t.}\quad\min\big(\mathtt{m}(-a,a),\mathtt{m}_+(a,c),\mathtt{m}_-(-a,c)\big)\geqslant\tfrac{1}{n}\Big\}.$$

The only difference is that in the alternative $(A2)$, one has to add, for instance in the case of the bifurcation in the parameter b,

- (Vanishing degeneracy) $\displaystyle\lim_{\mathtt{s}\to\pm\infty}b_{\mathbf{m}}(\mathtt{s},a,c)=a.$

But this situation can be included in the "Collision of the boundaries" alternative.

A Elements of bifurcation theory

The purpose of this appendix is to expose the theoretical bifurcation results used in this work. We first start by the classical local bifurcation theorem of Crandall-Rabinowitz [9], see also [30, p. 15]. The version presented here is in the analytic setting which fits more with our goal. We also add to the statement the required conditions to get a pitchfork-type bifurcation. For more details, we refer the reader to the works of Shi [38] and Liu-Shi [32].

Theorem A.1. ***(Analytic local bifurcation + pitchfork property)***
Let X and Y be two Banach spaces. Let $(\lambda_0,u_0)\in\mathbb{R}\times X$ and U be a neighborhood of (λ_0,u_0) in $\mathbb{R}\times X$. Consider a real-analytic function $F:U\to Y$ such that

$(L1)$ $\forall(\lambda,u_0)\in U,\quad F(\lambda,u_0)=0.$

(*L*2) *$d_uF(\lambda_0,u_0)$ is a Fredholm operator with*

$$\dim\Big(\ker\big(d_uF(\lambda_0,u_0)\big)\Big)=1=\operatorname{codim}\Big(R\big(d_uF(\lambda_0,u_0)\big)\Big),\qquad \ker\big(d_uF(\lambda_0,u_0)\big)=\mathtt{span}(w_0).$$

(*L*3) *Transversality:*

$$\partial_\lambda d_uF(\lambda_0,u_0)\notin R\big(d_uF(\lambda_0,u_0)\big).$$

If we decompose

$$X=\mathtt{span}(w_0)\oplus Z,$$

then there exist two real-analytic functions

$$\lambda:(-\delta,\delta)\to\mathbb{R}\qquad\text{and}\qquad z:(-\delta,\delta)\to Z,\qquad\text{with}\qquad\delta>0,$$

such that

$$\lambda(0)=\lambda_0,\qquad z(0)=0$$

and the set of zeros of F in U is the union of two curves

$$\big\{(\lambda,u)\in U\quad\text{s.t.}\quad F(\lambda,u)=0\big\}=\big\{(\lambda,u_0)\in U\big\}\cup\mathscr{C}_{\text{local}},\qquad \mathscr{C}_{\text{local}}\triangleq\big\{\big(\lambda(\mathtt{s}),u_0+\mathtt{s}w_0+\mathtt{s}z(\mathtt{s})\big),\quad|\mathtt{s}|<\delta\big\}.$$

Assume in addition that

$$d_u^2F(\lambda_0,u_0)[w_0,w_0]\in R\big(d_uF(\lambda_0,u_0)\big).$$

Then $\lambda'(0)=0$ and if we denote

$$R\big(d_uF(\lambda_0,u_0)\big)=\ker(l)\qquad\text{for some } l\in Y^*,$$

then we have

$$\lambda''(0)=\frac{3\big\langle l\,,\,d_u^2F(\lambda_0,u_0)[w_0,\theta_0]\big\rangle-\big\langle l\,,\,d_u^3F(\lambda_0,u_0)[w_0,w_0,w_0]\big\rangle}{3\big\langle l\,,\,\partial_\lambda d_uF(\lambda_0,u_0)\big\rangle},$$

where θ_0 is solution of

$$d_uF(\lambda_0,u_0)[\theta_0]=d_u^2F(\lambda_0,u_0)[w_0,w_0].$$

If $\lambda''(0)\neq0$, we say that the bifurcation is of pitchfork-type. More precisely, the condition $\lambda''(0)>0$ (resp. $\lambda''(0)<0$) is called supercritical (resp. subcritical) bifurcation.

Finally, we present the classical global bifurcation theorem of Dancer [10], Buffoni-Toland [4, Thm. 9.1.1]. The version given here is taken from [7, Thm. 4].

Theorem A.2. ***(Analytic global bifurcation)*** *Let X and Y be two Banach spaces. Let U be an open subset of $\mathbb{R}\times X$. Consider a real-analytic function $F:U\to Y$ satisfying (L1), (L2), (L3) and the following additional properties.*

(*G*1) *For any $(\lambda,u)\in U$ such that $F(\lambda,u)=0$, the operator $d_uF(\lambda,u)$ is a Fredholm operator of index 0.*

(*G*2) *Assume that we can write*

$$U=\bigcup_{n\in\mathbb{N}}F_n,$$

where for any $n\in\mathbb{N}$, the set F_n is bounded and closed in $\mathbb{R}\times X$. Suppose that for any $n\in\mathbb{N}$, the set

$$\mathscr{S}_n\triangleq\Big\{(\lambda,u)\in F_n\quad\text{s.t.}\quad F(\lambda,u)=0\Big\}$$

is compact in $\mathbb{R}\times X$.

Then there exists a unique (up to reparametrization) continuous curve $\mathscr{C}_{\text{global}}$ such that

$$\mathscr{C}_{\text{local}}\subset\mathscr{C}_{\text{global}}\triangleq\Big\{\big(\lambda(\mathtt{s}),u(\mathtt{s})\big),\quad\mathtt{s}\in\mathbb{R}\Big\}\subset U\cap F^{-1}\big(\{0\}\big).$$

Moreover, $\mathscr{C}_{\text{global}}$ admits locally around each of its points a real-analytic parametrization. In addition, one of the following alternatives occurs

(*A*1) *there exists $T>0$ such that*

$$\forall\mathtt{s}\in\mathbb{R},\quad\lambda(\mathtt{s}+T)=\lambda(\mathtt{s})\qquad\text{and}\qquad u(\mathtt{s}+T)=u(\mathtt{s}).$$

(*A*2) *for any $n\in\mathbb{N}$, there exists $\mathtt{s}_n>0$ such that*

$$\forall\mathtt{s}>\mathtt{s}_n,\quad\big(\lambda(\mathtt{s}),u(\mathtt{s})\big)\notin F_n.$$

B A useful lemma in linear algebra

We present here a simple lemma used in our analysis to describe the range of the linearized operator.

Lemma B.1. *Let E be a vector space over a field $\mathbb{K}$. Let V_1 and V_2 be two subspaces of E of codimension one in E and such that $V_2 \subset V_1$. Then $V_2 = V_1$.*

Proof. By definition, since V_1 and V_2 are of codimension one in E, then there exists $(u_1, u_2) \in E^2$ such that $u_1 \notin V_1$, $u_2 \notin V_2$ and

$$V_1 \oplus \langle u_1 \rangle = E = V_2 \oplus \langle u_2 \rangle. \tag{B.1}$$

Let $v_1 \in V_1$. According to (B.1), there exists $(v_2, \lambda_u) \in V_2 \times \mathbb{K}$ such that

$$v_1 = v_2 + \lambda_u u_2.$$

Assume for the sake of contradiction that $\lambda_u \neq 0$. Then, using the fact that V_1 is a vector space together with $v_1 \in V_1$ and $v_2 \in V_2 \subset V_1$, we have

$$u_2 = \lambda_u^{-1}(v_1 - v_2) \in V_1.$$

Combined with the hypothesis $V_2 \subset V_1$, we deduce that

$$V_2 \oplus \langle u_2 \rangle \subset V_1.$$

This contradicts (B.1). Thus, $\lambda_u = 0$ and $v_1 = v_2 \in V_2$. This achieves the proof of the lemma. □

We mention that this lemma is just a reformulation of the following classical result on hyperplanes/linear forms.

Lemma B.2. *Let E be a vector space over a field $\mathbb{K}$. Then,*

1. *The hyperplanes (subspaces of codimension one) of E are exactly the kernels of non-zero linear forms on E.*
2. *Let φ be a non-zero linear form on E. Then, any linear form vanishing on the hyperplane $\ker(\varphi)$ is proportional to φ.*
3. *Two non-zero linear forms on E have the same kernel if and only if they are proportional.*

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